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- ▶ Choose path with maximum bottleneck capacity.
- ▶ Choose path with sufficiently large bottleneck capacity.
- ▶ Choose the shortest augmenting path.

Capacity Scaling

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Intuition:

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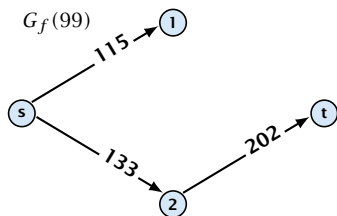
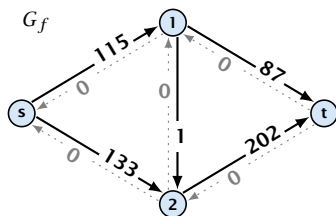
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- ▶ $G_f(\Delta)$ is a sub-graph of the residual graph G_f that contains only edges with capacity at least Δ .

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Algorithm 2 maxflow(G, s, t, c)

```
1: foreach  $e \in E$  do  $f_e \leftarrow 0$ ;  
2:  $\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$   
3: while  $\Delta \geq 1$  do  
4:    $G_f(\Delta) \leftarrow \Delta$ -residual graph  
5:   while there is augmenting path  $P$  in  $G_f(\Delta)$  do  
6:      $f \leftarrow \text{augment}(f, c, P)$   
7:      $\text{update}(G_f(\Delta))$   
8:    $\Delta \leftarrow \Delta/2$   
9: return  $f$ 
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- ▶ therefore after the last phase there are no augmenting paths anymore
- ▶ this means we have a maximum flow.

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- ▶ There must exist an s - t cut in $G_f(\Delta)$ of zero capacity.
- ▶ In G_f this cut can have capacity at most $m\Delta$.
- ▶ This gives me an upper bound on the flow that I can still add.

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Theorem 4

We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.