

# Baseball Elimination

<i>team</i> <i>i</i>	<i>wins</i> $w_i$	<i>losses</i> $\ell_i$	<i>remaining games</i>			
			<i>Atl</i>	<i>Phi</i>	<i>NY</i>	<i>Mon</i>
Atlanta	83	71	–	1	6	1
Philadelphia	80	79	1	–	0	2
New York	78	78	6	0	–	0
Montreal	77	82	1	2	0	–

**Which team can end the season with most wins?**

- ▶ Montreal is eliminated, since even after winning all remaining games there are only 80 wins.
- ▶ But also Philadelphia is eliminated. Why?

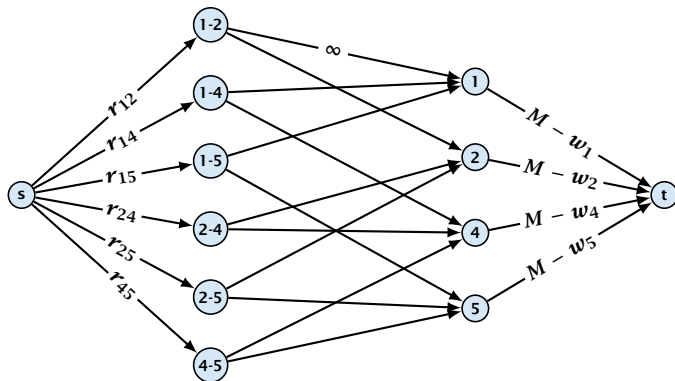
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## Formal definition of the problem:

- ▶ Given a set  $S$  of teams, and one specific team  $z \in S$ .
- ▶ Team  $x$  has already won  $w_x$  games.
- ▶ Team  $x$  still has to play team  $y$ ,  $r_{xy}$  times.
- ▶ Does team  $z$  still have a chance to finish with the most number of wins.

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Flow network for  $z = 3$ .  $M$  is number of wins Team 3 can still obtain.

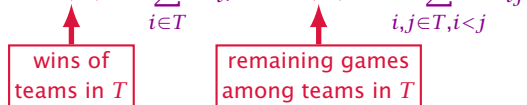


**Idea.** Distribute the results of remaining games in such a way that no team gets too many wins.

# Certificate of Elimination

Let  $T \subseteq S$  be a subset of teams. Define

$$w(T) := \sum_{i \in T} w_i, \quad r(T) := \sum_{i, j \in T, i < j} r_{ij}$$



If  $\frac{w(T)+r(T)}{|T|} > M$  then one of the teams in  $T$  will have more than  $M$  wins in the end. A team that can win at most  $M$  games is therefore eliminated.

## Theorem 1

A team  $z$  is eliminated if and only if the flow network for  $z$  does not allow a flow of value  $\sum_{i,j \in S \setminus \{z\}, i < j} r_{ij}$ .

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$$r(S \setminus \{z\})$$



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$$\begin{aligned} r(S \setminus \{z\}) &> \text{cap}(A, V \setminus A) \\ &\geq \sum_{i < j: i \notin T \vee j \notin T} r_{ij} + \sum_{i \in T} (M - w_i) \end{aligned}$$

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- ▶ This gives  $M < (w(T) + r(T))/|T|$ , i.e.,  $z$  is eliminated.

# Baseball Elimination

## Proof ( $\Rightarrow$ )

- ▶ Suppose we have a flow that saturates all source edges.
- ▶ We can assume that this flow is *integral*.
- ▶ For every pairing  $x$ - $y$  it defines how many games team  $x$  and team  $y$  should win.
- ▶ The flow leaving the team-node  $x$  can be interpreted as the additional number of wins that team  $x$  will obtain.
- ▶ This is less than  $M - w_x$  because of capacity constraints.
- ▶ Hence, we found a set of results for the remaining games, such that no team obtains more than  $M$  wins in total.
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