

# 7 Dictionary

## Dictionary:

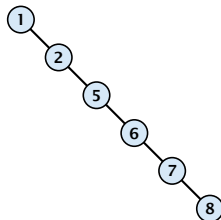
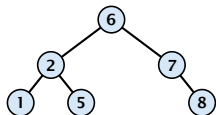
- ▶  **$S$ . insert( $x$ )**: Insert an element  $x$ .
- ▶  **$S$ . delete( $x$ )**: Delete the element pointed to by  $x$ .
- ▶  **$S$ . search( $k$ )**: Return a pointer to an element  $e$  with  $\text{key}[e] = k$  in  $S$  if it exists; otherwise return **null**.

## 7.1 Binary Search Trees

An (**internal**) **binary search tree** stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node  $v$  have a smaller key-value than  $\text{key}[v]$  and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(**External** Search Trees store objects only at leaf-vertices)

Examples:

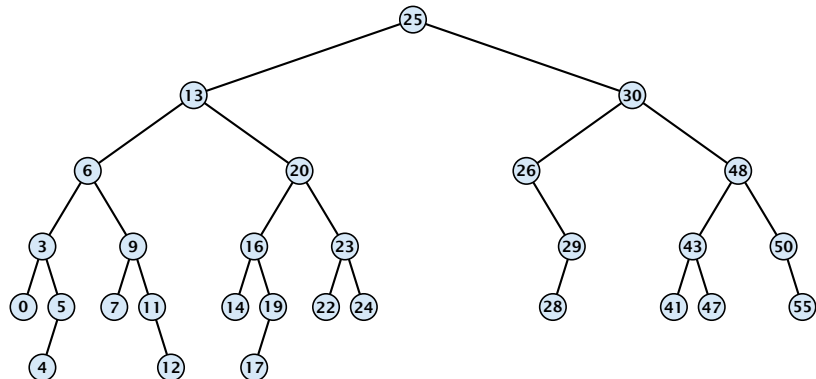


## 7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- ▶  $T.\text{insert}(x)$
- ▶  $T.\text{delete}(x)$
- ▶  $T.\text{search}(k)$
- ▶  $T.\text{successor}(x)$
- ▶  $T.\text{predecessor}(x)$
- ▶  $T.\text{minimum}()$
- ▶  $T.\text{maximum}()$

# Binary Search Trees: Searching

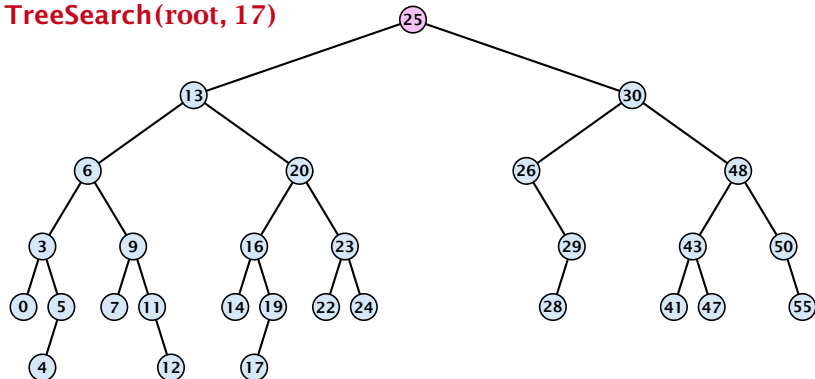


## Algorithm 1 TreeSearch( $x, k$ )

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TreeSearch(root, 17)

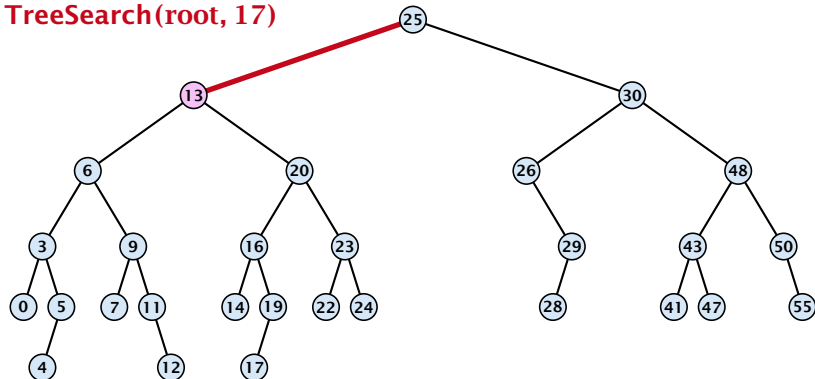


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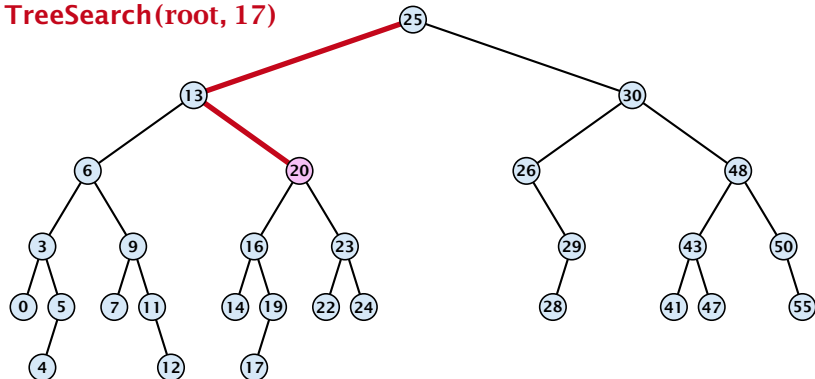


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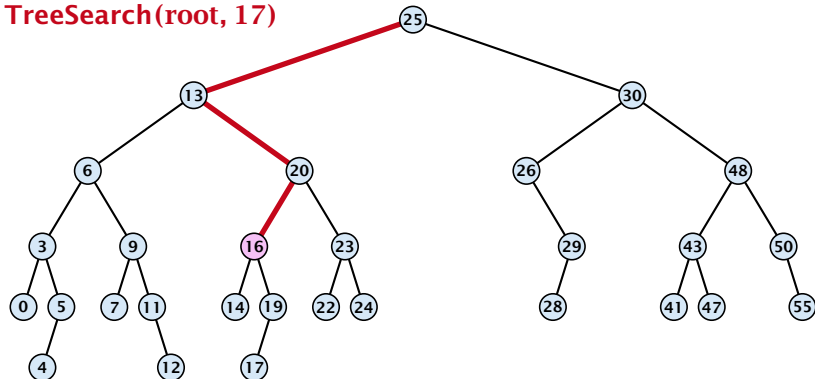


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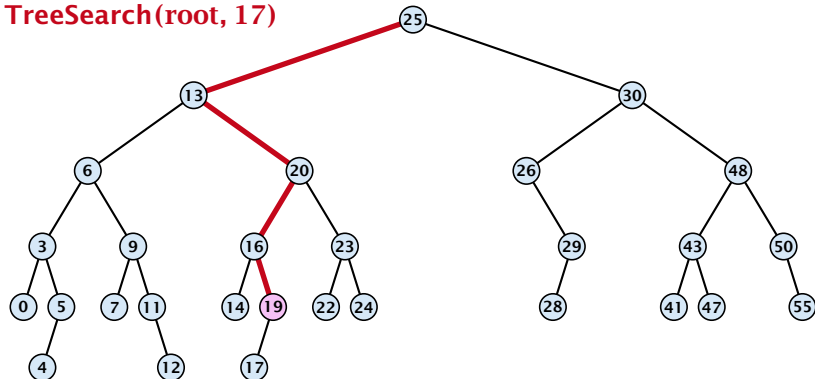
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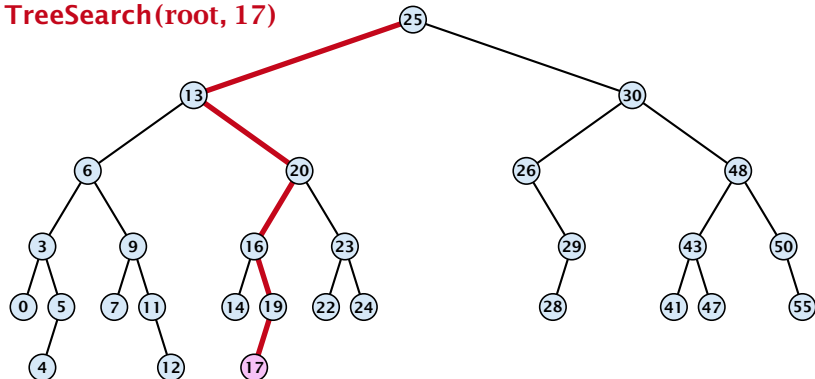


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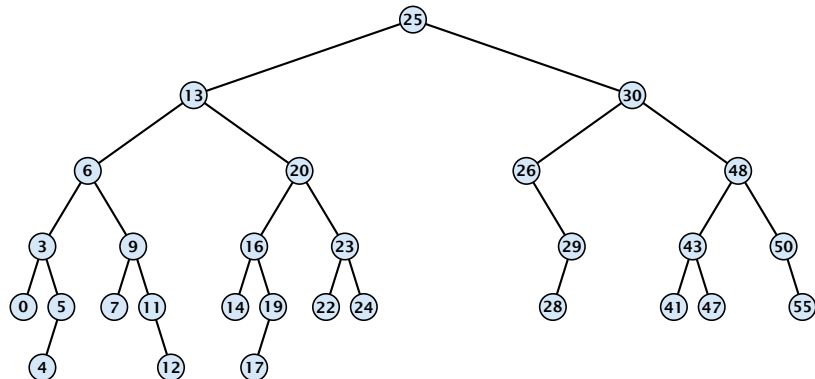
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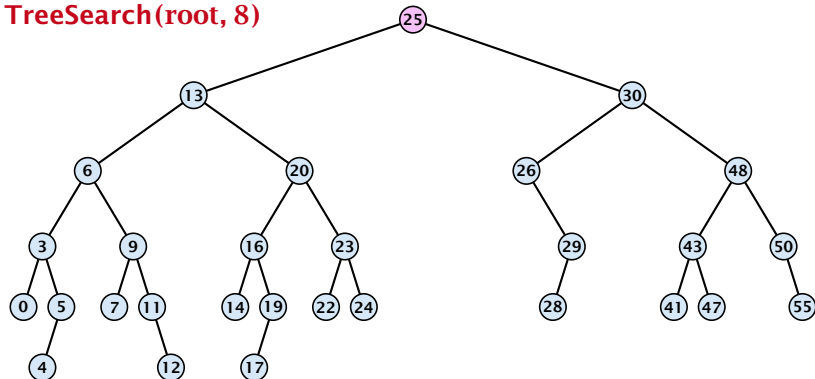


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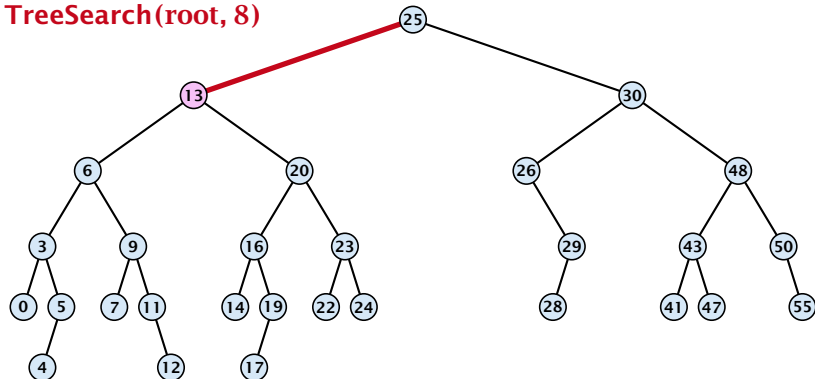


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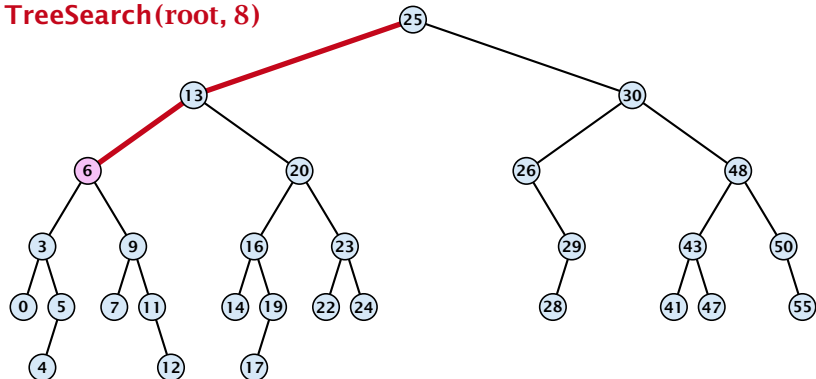


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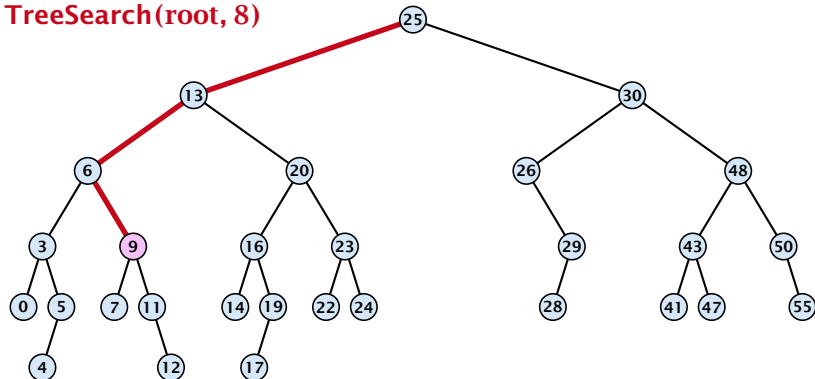


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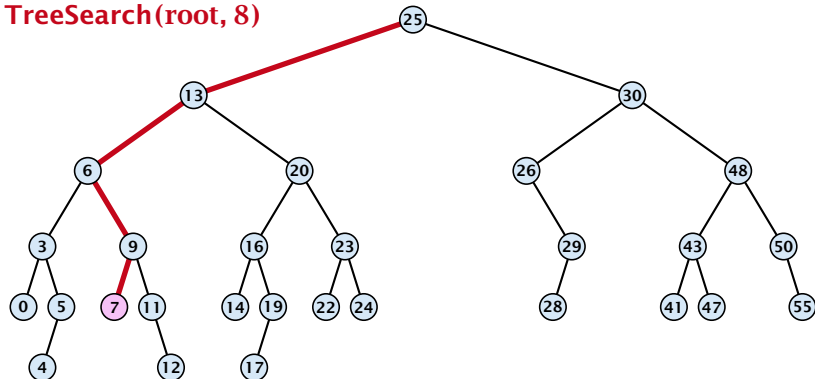


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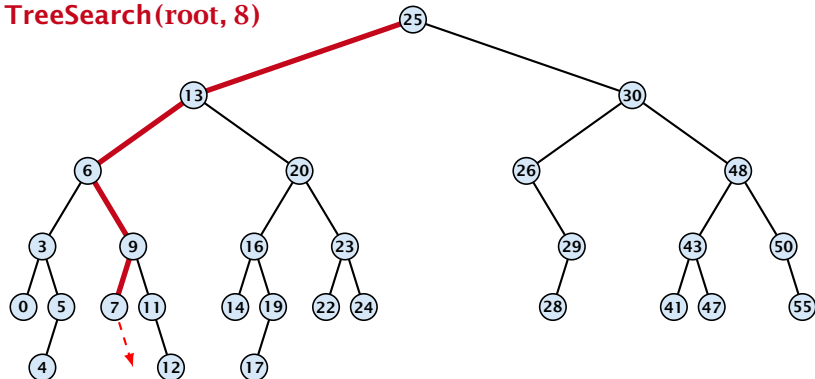
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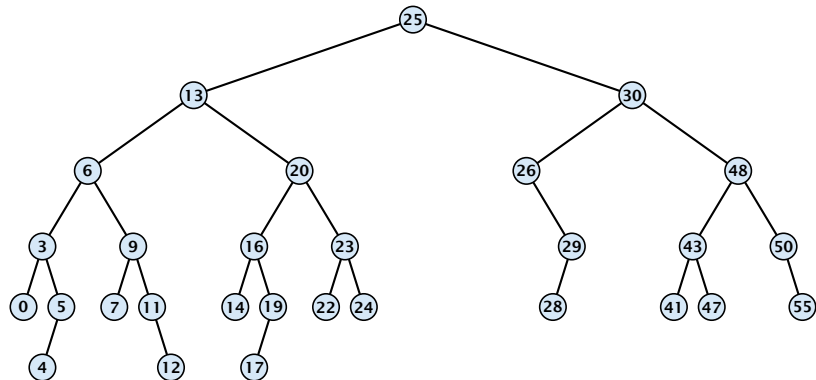
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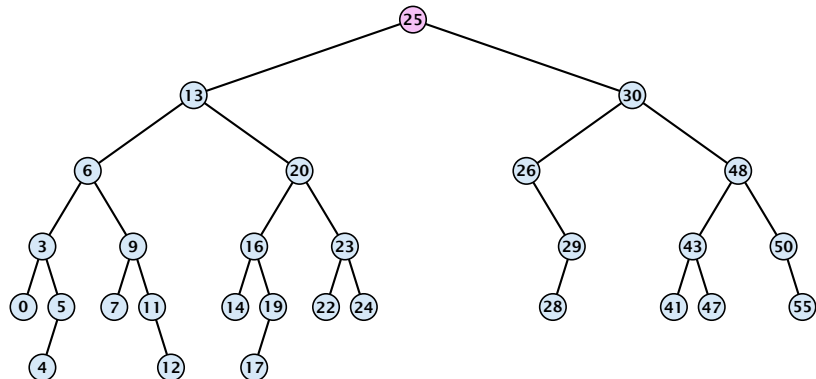
# Binary Search Trees: Minimum



## Algorithm 2 TreeMin( $x$ )

- 1: **if**  $x = \text{null}$  **or**  $\text{left}[x] = \text{null}$  **return**  $x$
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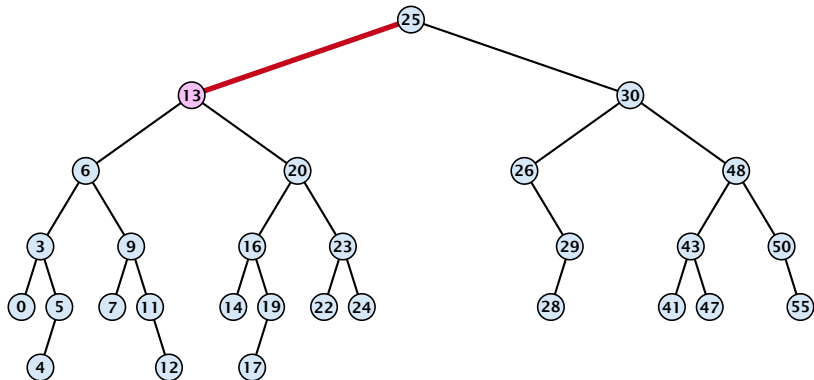
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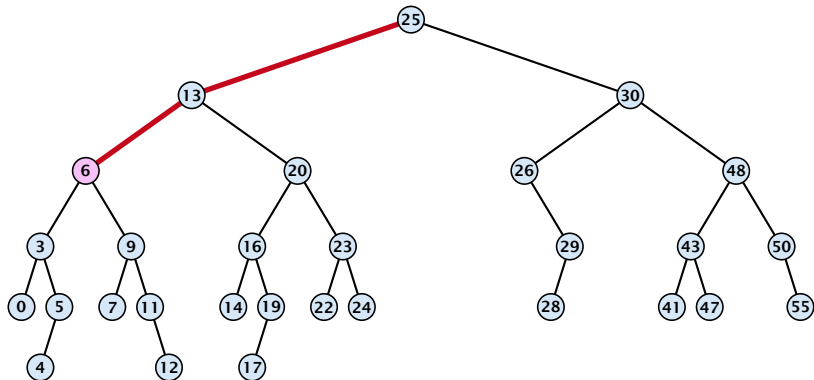
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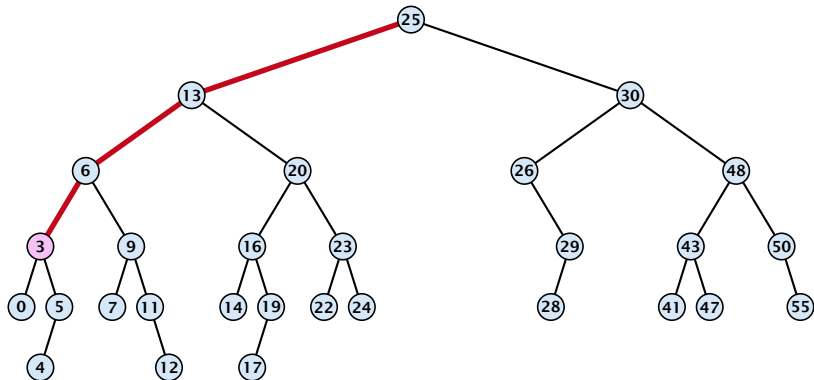
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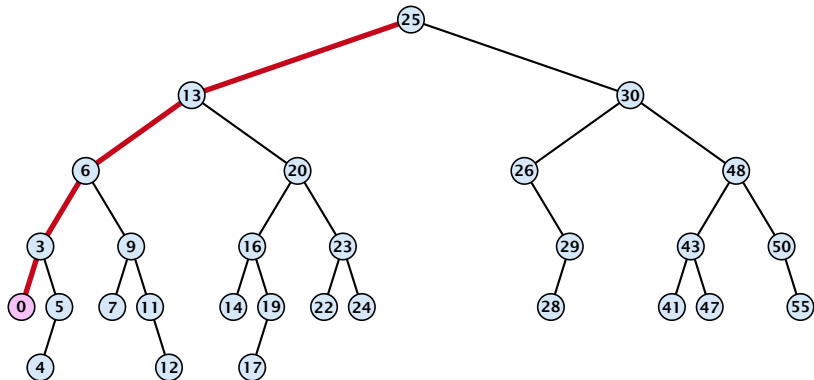
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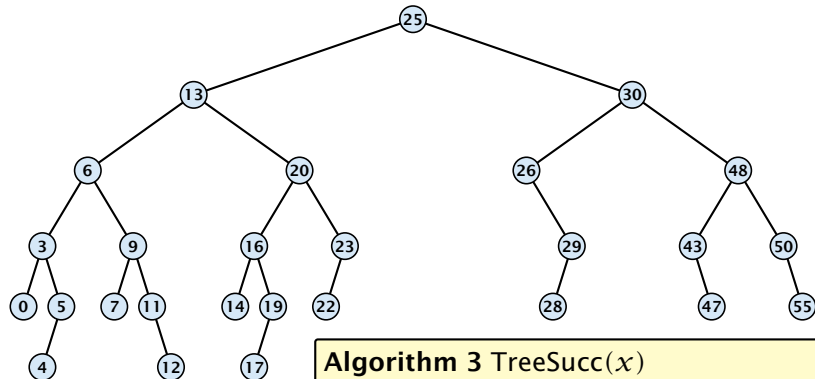
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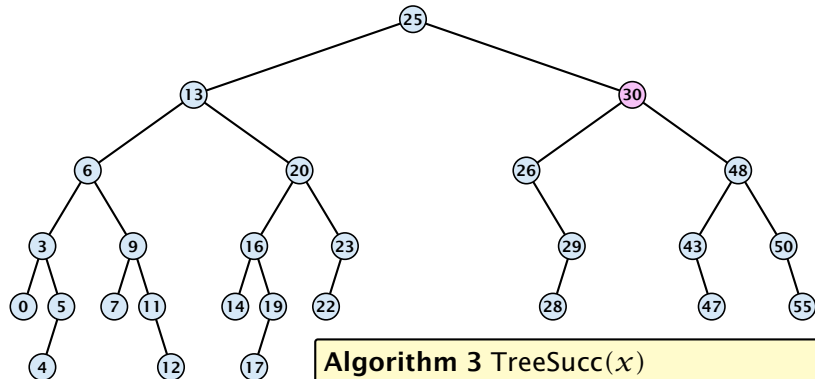


## Algorithm 3 TreeSucc( $x$ )

- 1: **if** right[ $x$ ]  $\neq$  null **return** TreeMin(right[ $x$ ])
- 2:  $y \leftarrow$  parent[ $x$ ]
- 3: **while**  $y \neq$  null **and**  $x =$  right[ $y$ ] **do**
- 4:      $x \leftarrow y$ ;  $y \leftarrow$  parent[ $x$ ]
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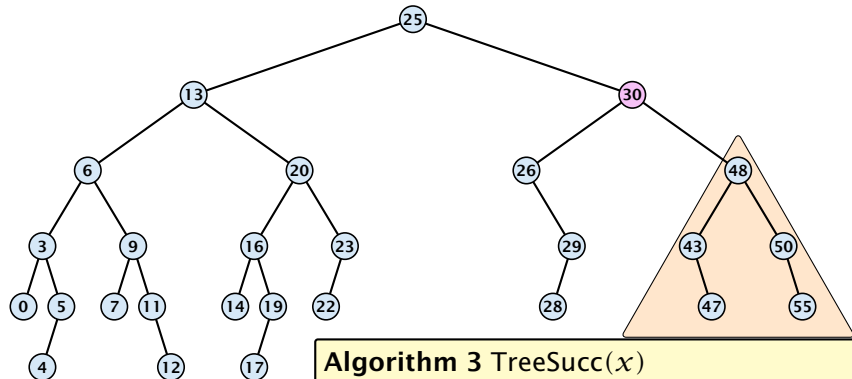
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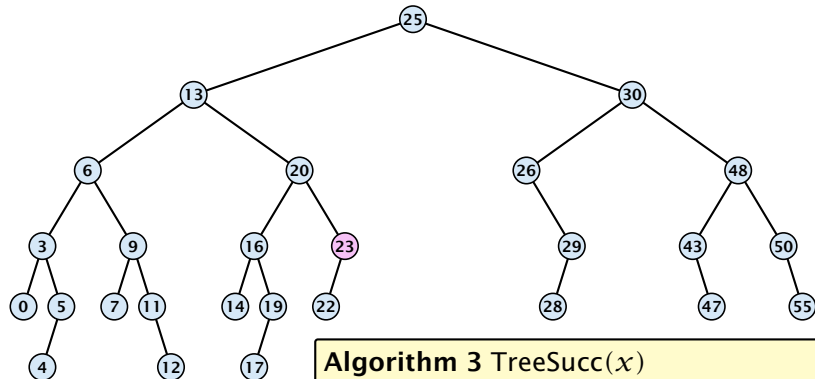
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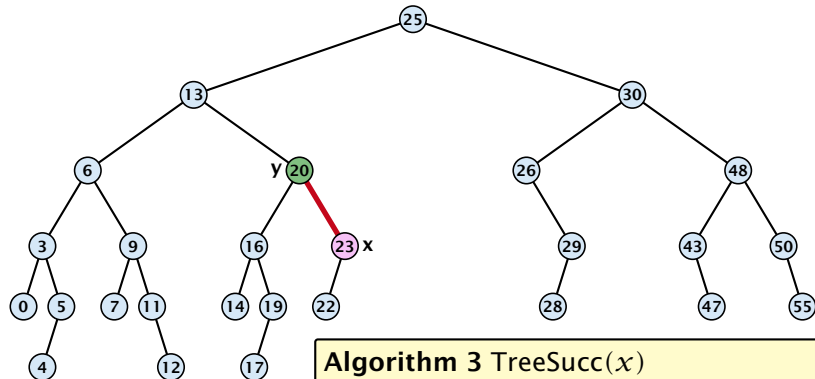
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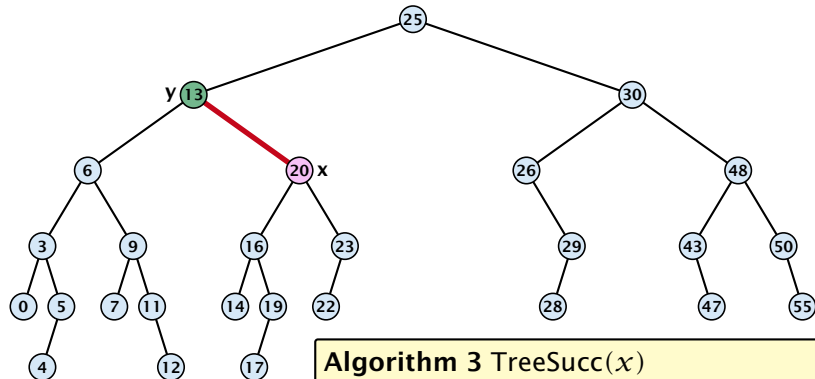
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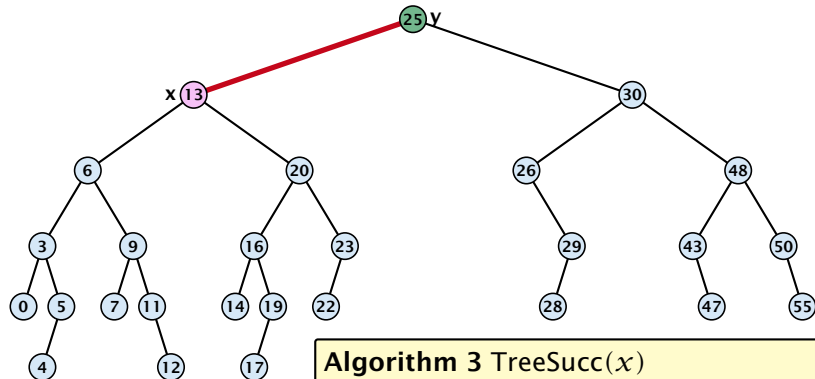
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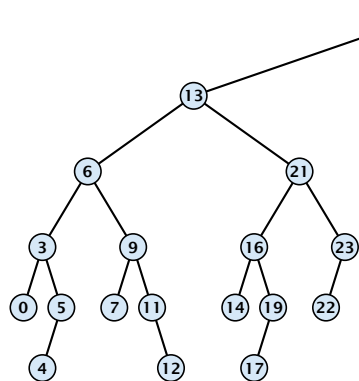
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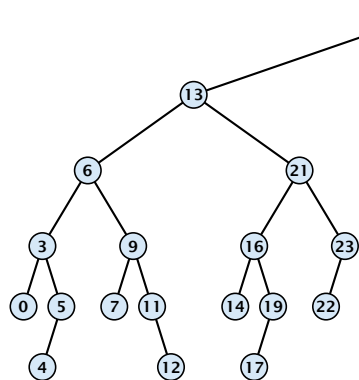


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1: if  $x = \text{null}$  then
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3:   return;
4: if key[ $x$ ] > key[ $z$ ] then
5:   if left[ $x$ ] = null then
6:     left[ $x$ ]  $\leftarrow z$ ; parent[ $z$ ]  $\leftarrow x$ ;
7:   else TreeInsert(left[ $x$ ],  $z$ );
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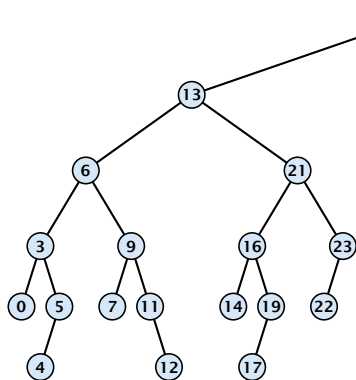
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Search for  $z$ . At some point the search stops at a null-pointer. This is the place to insert  $z$ .

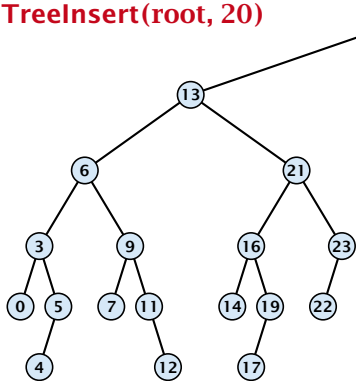
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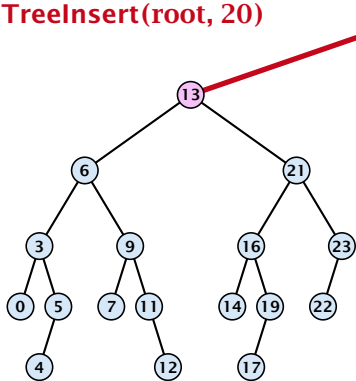
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- 3:     **return**;
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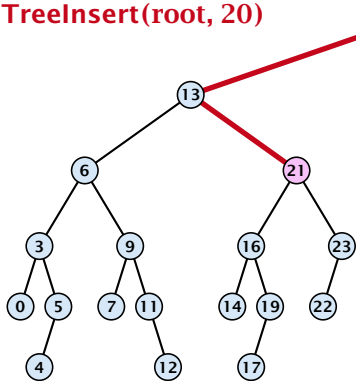
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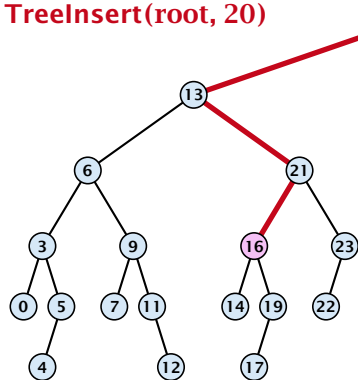
## Algorithm 4 TreeInsert( $x, z$ )

```
1: if  $x = \text{null}$  then
2:   root[ $T$ ]  $\leftarrow z$ ; parent[ $z$ ]  $\leftarrow \text{null}$ ;
3:   return;
4: if key[ $x$ ] > key[ $z$ ] then
5:   if left[ $x$ ] = null then
6:     left[ $x$ ]  $\leftarrow z$ ; parent[ $z$ ]  $\leftarrow x$ ;
7:   else TreeInsert(left[ $x$ ],  $z$ );
8: else
9:   if right[ $x$ ] = null then
10:    right[ $x$ ]  $\leftarrow z$ ; parent[ $z$ ]  $\leftarrow x$ ;
11:   else TreeInsert(right[ $x$ ],  $z$ );
```

# Binary Search Trees: Insert

Insert element **not** in the tree.

**TreeInsert**(root, 20)



Search for  $z$ . At some point the search stops at a null-pointer. This is the place to insert  $z$ .

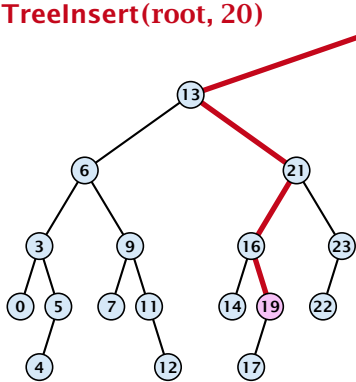
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- 3:     **return**;
- 4: **if**  $\text{key}[x] > \text{key}[z]$  **then**
- 5:     **if**  $\text{left}[x] = \text{null}$  **then**
- 6:          $\text{left}[x] \leftarrow z$ ;  $\text{parent}[z] \leftarrow x$ ;
- 7:     **else** TreeInsert( $\text{left}[x], z$ );
- 8: **else**
- 9:     **if**  $\text{right}[x] = \text{null}$  **then**
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- 11:     **else** TreeInsert( $\text{right}[x], z$ );

# Binary Search Trees: Insert

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**TreeInsert**(root, 20)



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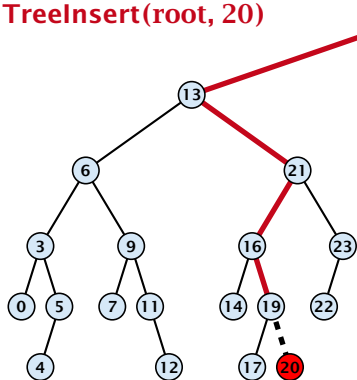
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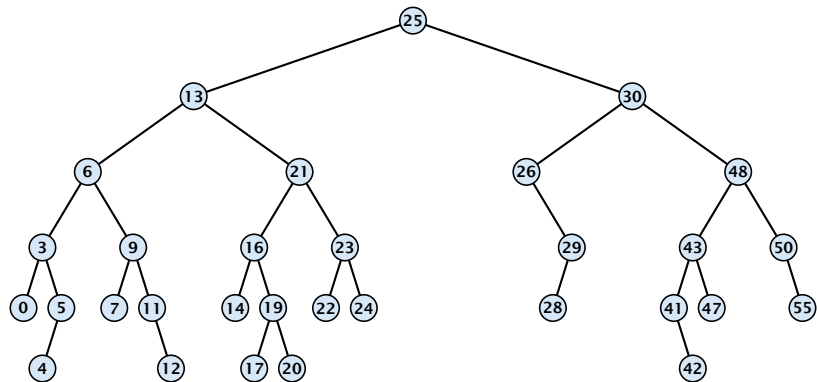


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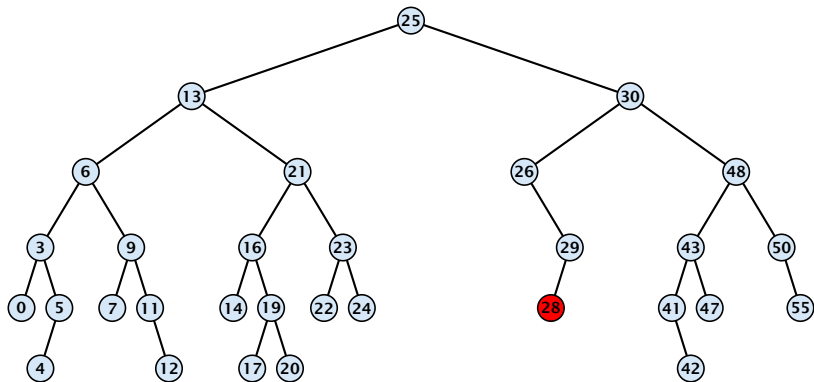
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# Binary Search Trees: Delete





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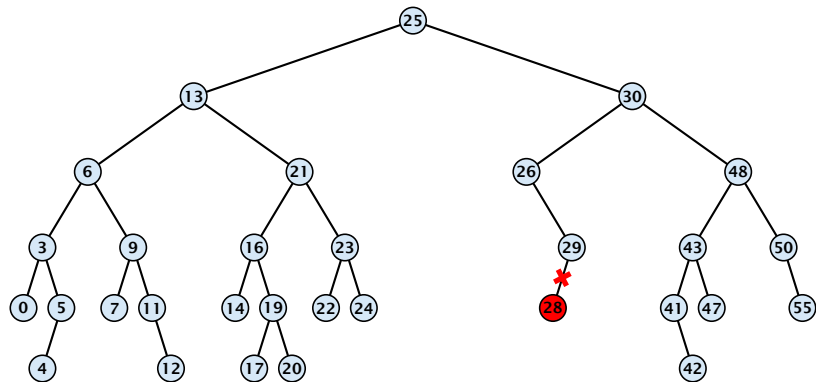


## Case 1:

Element does not have any children

- ▶ Simply go to the parent and set the corresponding pointer to **null**.

# Binary Search Trees: Delete

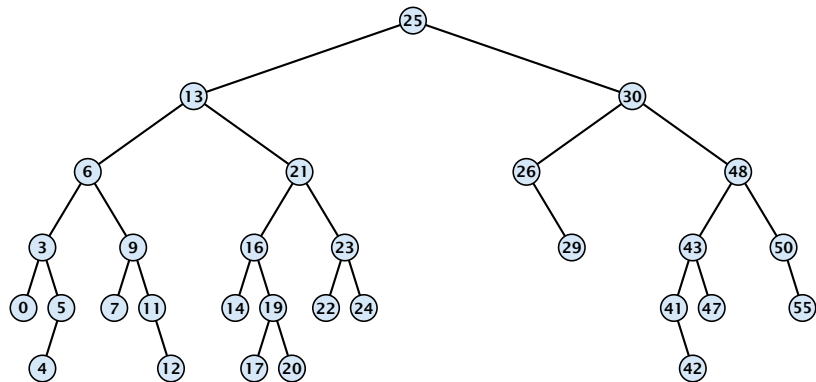


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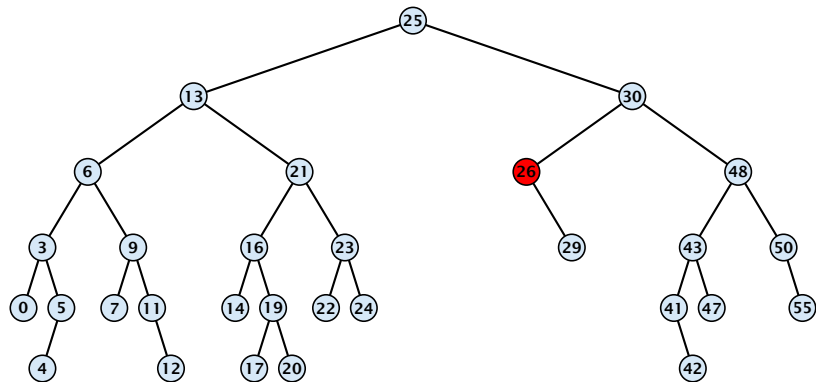


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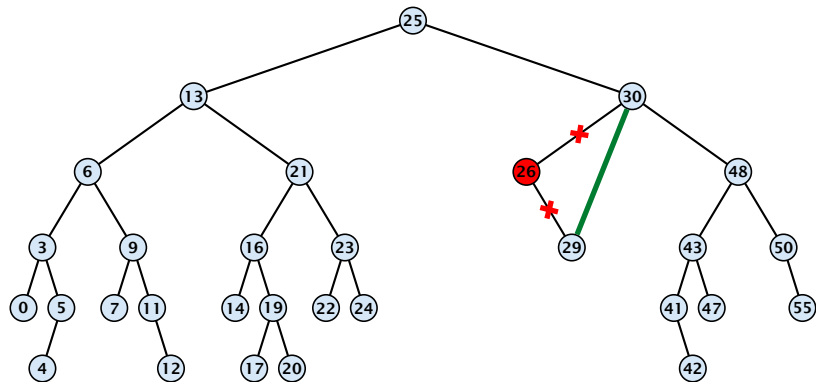


### Case 2:

Element has exactly one child

- ▶ Splice the element out of the tree by connecting its parent to its successor.

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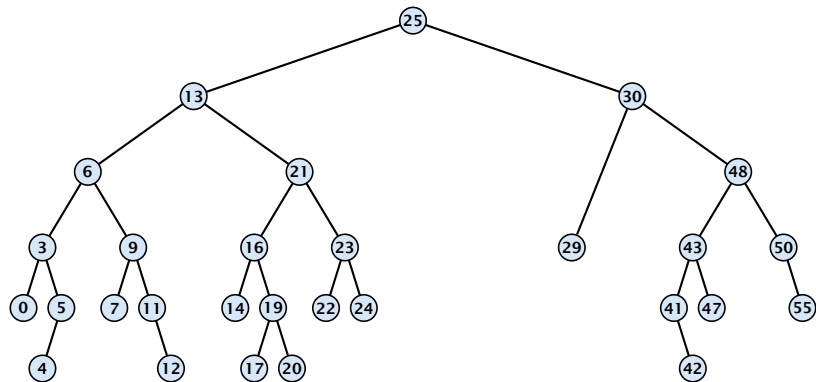


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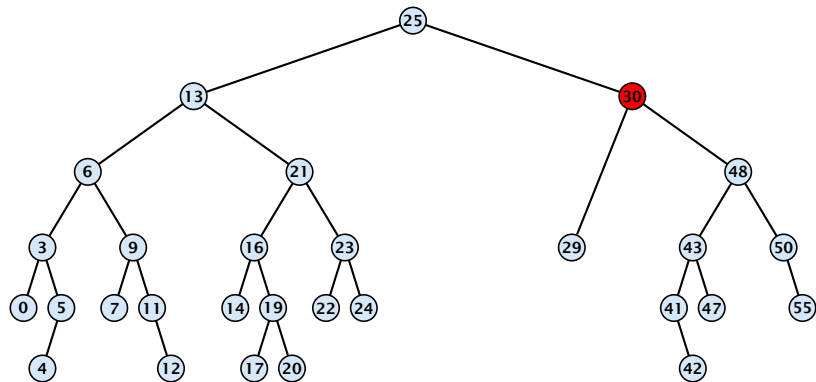


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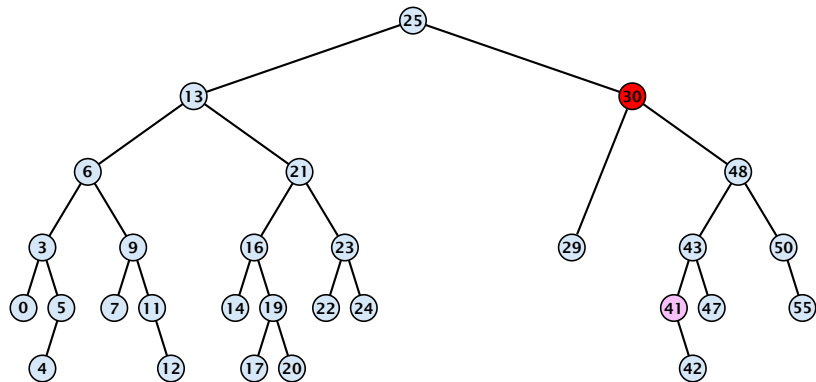


## Case 3:

Element has two children

- ▶ Find the successor of the element
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- ▶ Replace content of element by content of successor

## Binary Search Trees: Delete



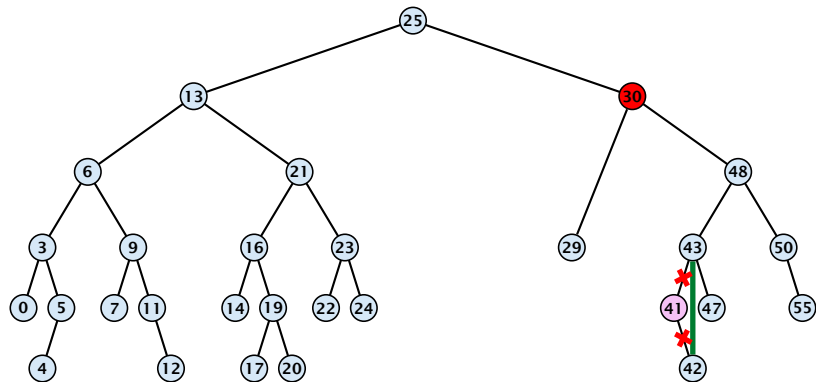
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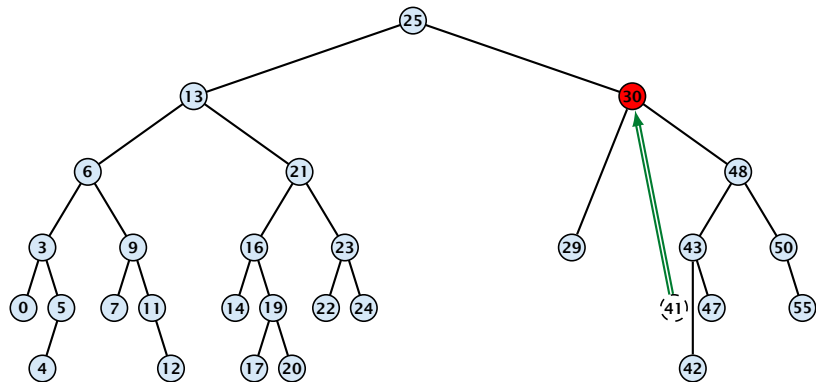


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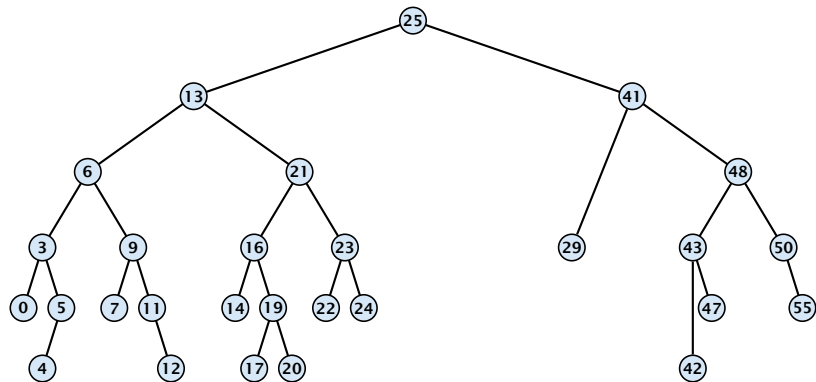


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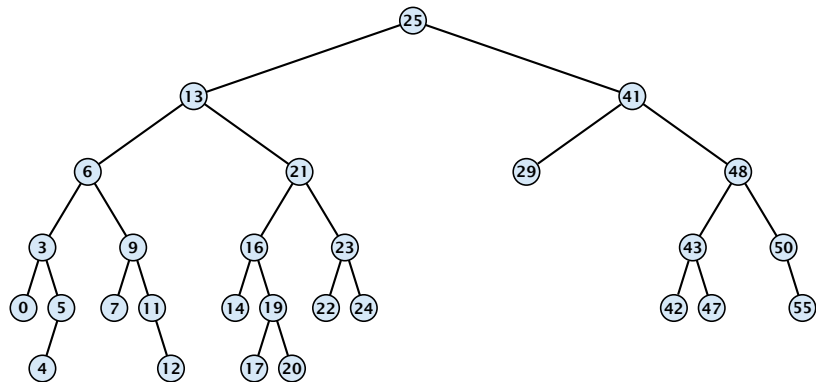


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- ▶ Splice successor out of the tree
- ▶ Replace content of element by content of successor

# Binary Search Trees: Delete

## Algorithm 9 TreeDelete( $z$ )

```
1: if left[ $z$ ] = null or right[ $z$ ] = null
2:   then  $y \leftarrow z$  else  $y \leftarrow \text{TreeSucc}(z)$ ;   select  $y$  to splice out
3: if left[ $y$ ]  $\neq$  null
4:   then  $x \leftarrow \text{left}[y]$  else  $x \leftarrow \text{right}[y]$ ;  $x$  is child of  $y$  (or null)
5: if  $x \neq \text{null}$  then parent[ $x$ ]  $\leftarrow$  parent[ $y$ ];   parent[ $x$ ] is correct
6: if parent[ $y$ ] = null then
7:   root[ $T$ ]  $\leftarrow x$ 
8: else
9:   if  $y = \text{left}[\text{parent}[y]]$  then
10:    left[parent[ $y$ ]]  $\leftarrow x$ 
11:   else
12:    right[parent[ $y$ ]]  $\leftarrow x$ 
13: if  $y \neq z$  then copy  $y$ -data to  $z$ 
```

} fix pointer to  $x$

# Balanced Binary Search Trees

All operations on a binary search tree can be performed in time  $\mathcal{O}(h)$ , where  $h$  denotes the height of the tree.

However the height of the tree may become as large as  $\Theta(n)$ .

## Balanced Binary Search Trees

With each insert- and delete-operation perform local adjustments to guarantee a height of  $\mathcal{O}(\log n)$ .

AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.

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## 7.2 Red Black Trees

### Definition 1

A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

1. The root is black.
2. All leaf nodes are black.
3. For each node, all paths to descendant leaves contain the same number of black nodes.
4. If a node is red then both its children are black.

The null-pointers in a binary search tree are replaced by pointers to special null-vertices, that do not carry any object-data

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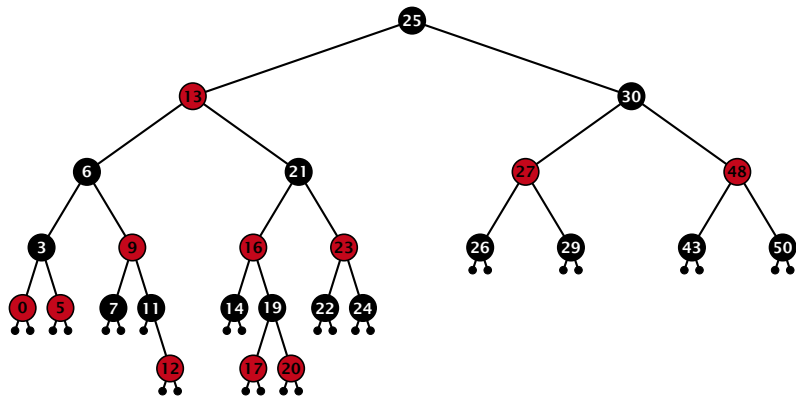
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# Red Black Trees: Example



## 7.2 Red Black Trees

### Lemma 2

A red-black tree with  $n$  internal nodes has height at most  $\mathcal{O}(\log n)$ .

### Definition 3

The **black height**  $\text{bh}(v)$  of a node  $v$  in a red black tree is the number of black nodes on a path from  $v$  to a leaf vertex (not counting  $v$ ).

We first show:

### Lemma 4

A sub-tree of black height  $\text{bh}(v)$  in a red black tree contains at least  $2^{\text{bh}(v)} - 1$  internal vertices.

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## 7.2 Red Black Trees

### Proof of Lemma 4.

Induction on the height of  $v$ .

base case ( $\text{height}(v) = 0$ )

if  $\text{height}(v)$  (maximum distance from  $v$  and a node in the subtree rooted at  $v$ ) is 0 then  $v$  is a leaf.

The black height of  $v$  is 0.

The subtree rooted at  $v$  contains only  $v$  and is therefore

balanced.

## 7.2 Red Black Trees

### Proof of Lemma 4.

#### Induction on the height of $v$ .

base case ( $\text{height}(v) = 0$ )

if  $b(v) > 0$ , maximum distance from  $v$  to a node in the subtree rooted at  $v$  is  $2b(v) - 1$ .

The black height of  $v$  is  $b(v)$ .

The subtree rooted at  $v$  contains  $2^{2b(v)} - 1$  nodes.

□

## 7.2 Red Black Trees

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- ▶ The sub-tree rooted at  $v$  contains  $0 = 2^{\text{bh}(v)} - 1$  inner vertices.

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## 7.2 Red Black Trees

### Proof (cont.)

#### induction step

- Suppose  $T$  is a node with height  $h$ .
  - If  $T$  has no children with strictly smaller height, then  $T$  is a leaf and the claim is true.
  - These children (if any) either have height  $h-1$  or  $h-2$ .
    - By induction hypothesis both sub-trees contain at least  $\frac{1}{2} \cdot 2^{h-1}$  internal vertices.
    - Their total is at least  $2^{h-1}$ .



## 7.2 Red Black Trees

### Proof (cont.)

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- ▶ Suppose  $v$  is a node with  $\text{height}(v) > 0$ .
- ▶  $v$  has two children with strictly smaller height.
- ▶ These children ( $c_1, c_2$ ) either have  $\text{bh}(c_i) = \text{bh}(v)$  or  $\text{bh}(c_i) = \text{bh}(v) - 1$ .
- ▶ By induction hypothesis both sub-trees contain at least  $2^{\text{bh}(v)-1} - 1$  internal vertices.
- ▶ Then  $T_v$  contains at least  $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$  vertices.



## 7.2 Red Black Trees

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Let  $h$  denote the height of the red-black tree, and let  $P$  denote a path from the root to the furthest leaf.

At least half of the nodes on  $P$  must be black, since a red node must be followed by a black node.

Hence, the black height of the root is at least  $h/2$ .

The tree contains at least  $2^{h/2} - 1$  internal vertices. Hence,  
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Hence,  $h \leq 2 \log(n + 1) = \mathcal{O}(\log n)$ . □



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Hence,  $h \leq 2 \log(n + 1) = \mathcal{O}(\log n)$ . □

## 7.2 Red Black Trees

### Proof of Lemma 2.

Let  $h$  denote the height of the red-black tree, and let  $P$  denote a path from the root to the furthest leaf.

At least half of the nodes on  $P$  must be black, since a red node must be followed by a black node.

Hence, the black height of the root is at least  $h/2$ .

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## 7.2 Red Black Trees

### Definition 1

A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

1. The root is black.
2. All leaf nodes are black.
3. For each node, all paths to descendant leaves contain the same number of black nodes.
4. If a node is red then both its children are black.

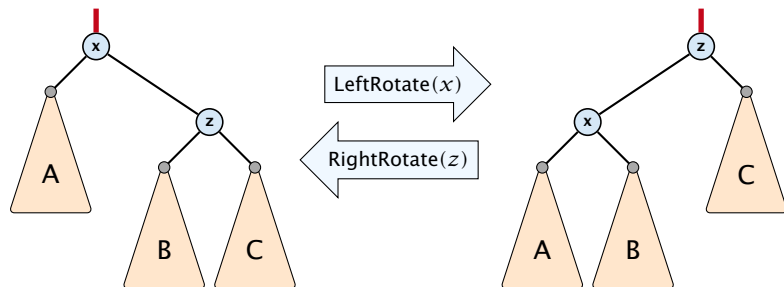
The **null**-pointers in a binary search tree are replaced by pointers to special null-vertices, that do not carry any object-data.

## 7.2 Red Black Trees

We need to adapt the insert and delete operations so that the red black properties are maintained.

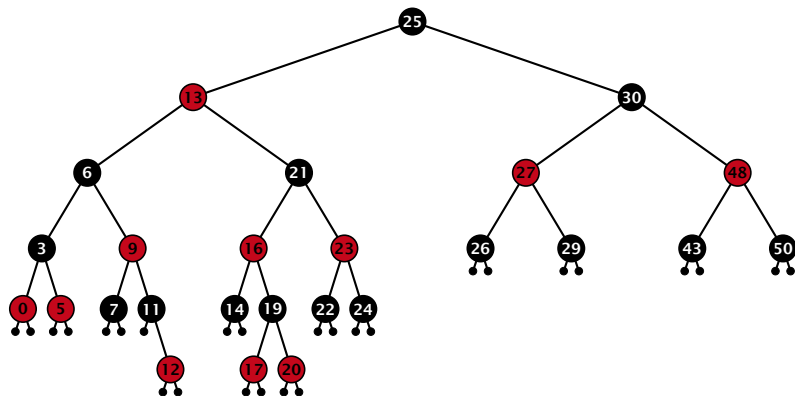
# Rotations

The properties will be maintained through rotations:





# Red Black Trees: Insert

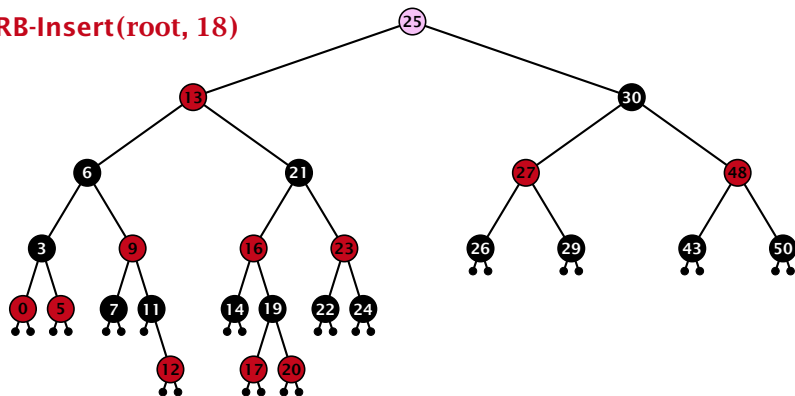


## Insert:

- ▶ first make a normal insert into a binary search tree
- ▶ then fix red-black properties

# Red Black Trees: Insert

RB-Insert(root, 18)

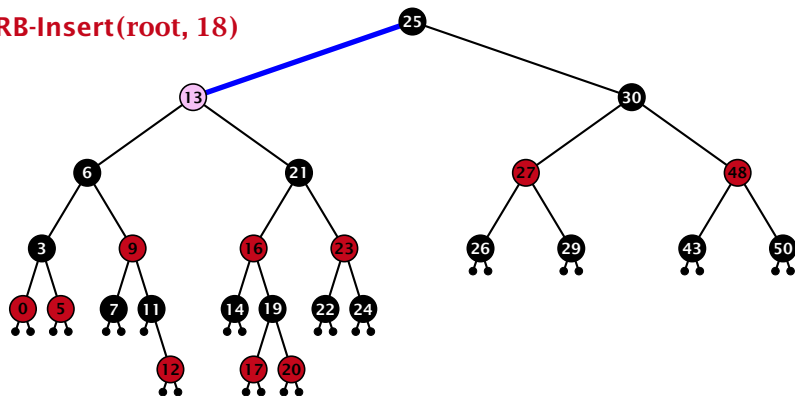


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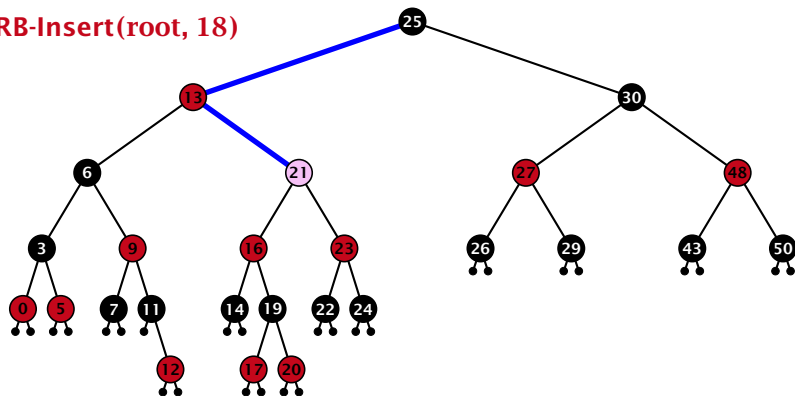


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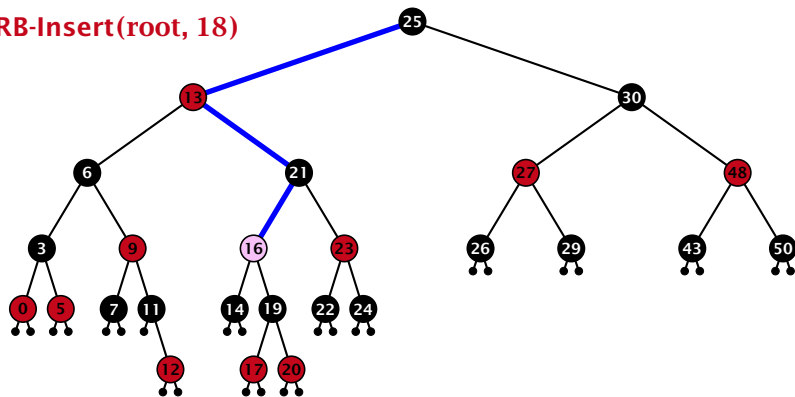


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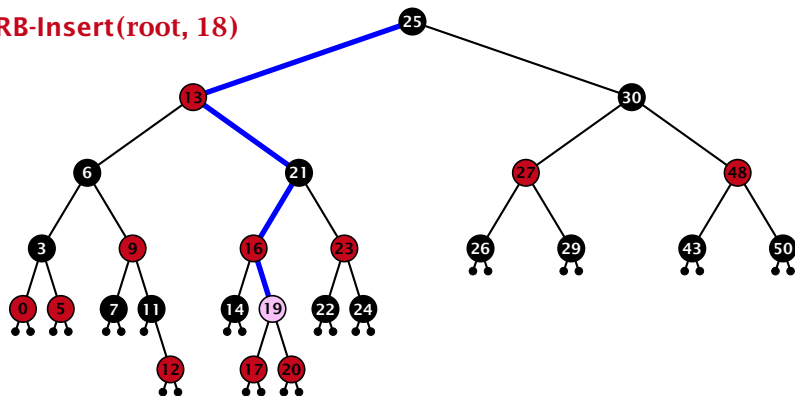


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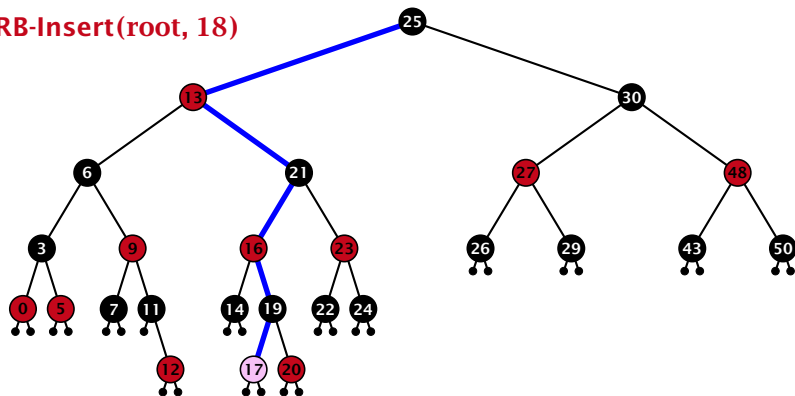


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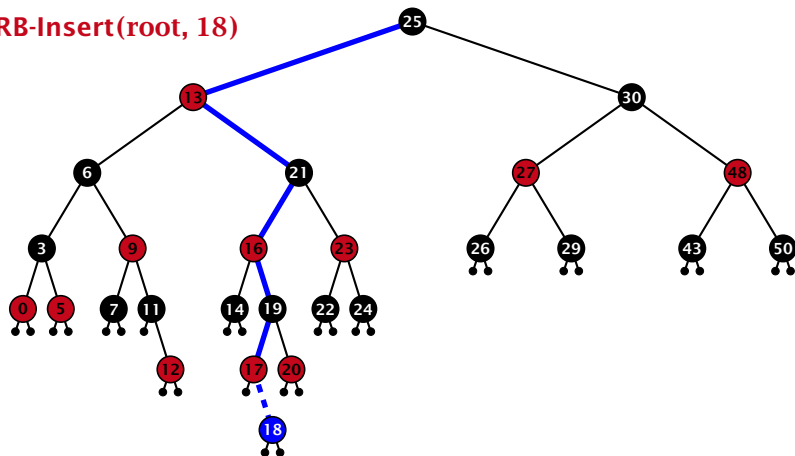


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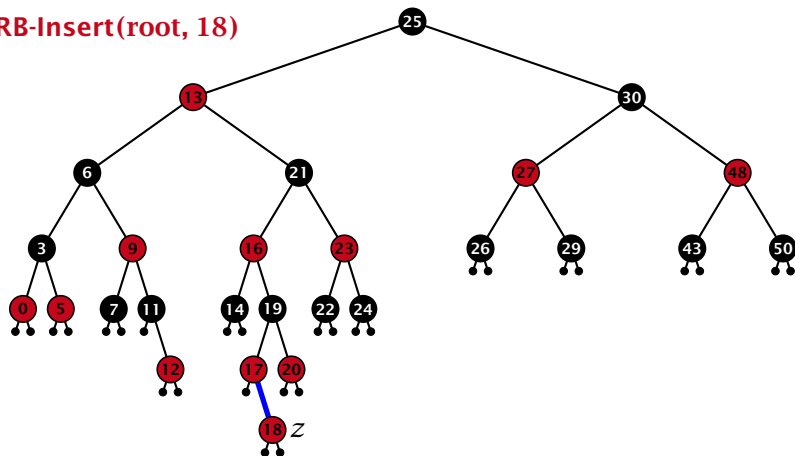
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## Invariant of the fix-up algorithm:

- ▶  $z$  is a red node
- ▶ the black-height property is fulfilled at every node
- ▶ the only violation of red-black properties occurs at  $z$  and  $\text{parent}[z]$ 
  - ▶ either both of them are red (most important case)
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If  $z$  has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

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# Red Black Trees: Insert

## Algorithm 10 InsertFix( $z$ )

```
1: while parent[ $z$ ]  $\neq$  null and col[parent[ $z$ ]] = red do
2:   if parent[ $z$ ] = left[gp[ $z$ ]] then
3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
5:       col[p[ $z$ ]]  $\leftarrow$  black; col[ $u$ ]  $\leftarrow$  black;
6:       col[gp[ $z$ ]]  $\leftarrow$  red;  $z \leftarrow$  grandparent[ $z$ ];
7:     else
8:       if  $z$  = right[parent[ $z$ ]] then
9:          $z \leftarrow$  p[ $z$ ]; LeftRotate( $z$ );
10:      col[p[ $z$ ]]  $\leftarrow$  black; col[gp[ $z$ ]]  $\leftarrow$  red;
11:      RightRotate(gp[ $z$ ]);
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13: col(root[ $T$ ])  $\leftarrow$  black;
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3:      $uncle \leftarrow$  right[grandparent[ $z$ ]]
4:     if col[ $uncle$ ] = red then
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7:   else Case 2: uncle black
8:     if  $z$  = right[parent[ $z$ ]] then
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## Red Black Trees: Insert

### Algorithm 10 InsertFix( $z$ )

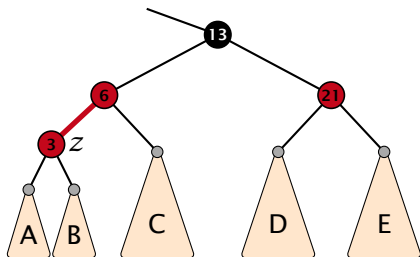
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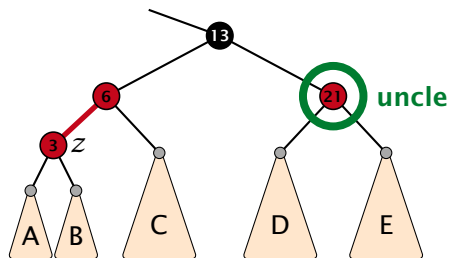
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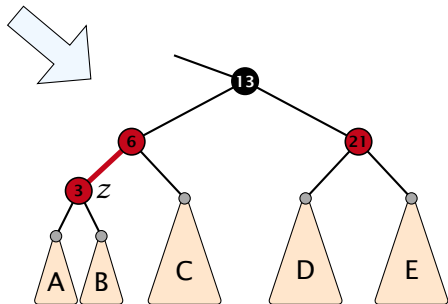
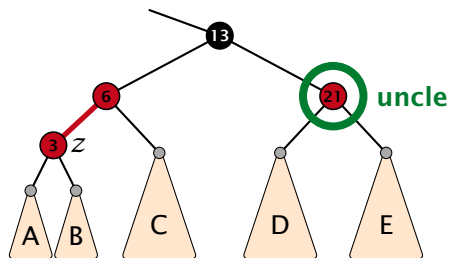
## Case 1: Red Uncle



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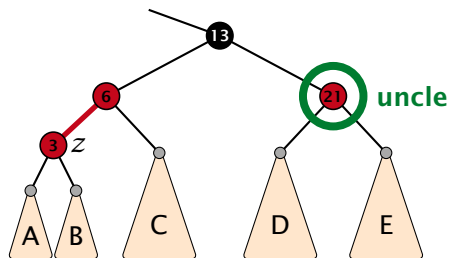


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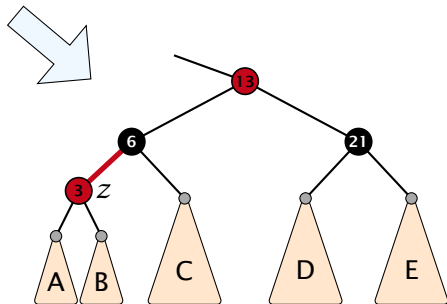




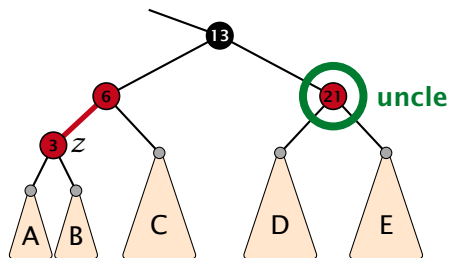
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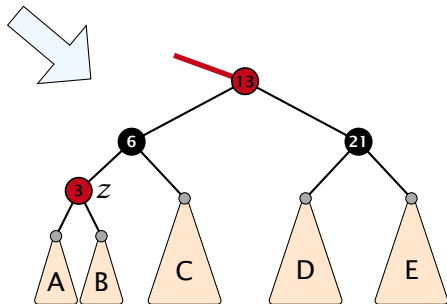
1. recolour



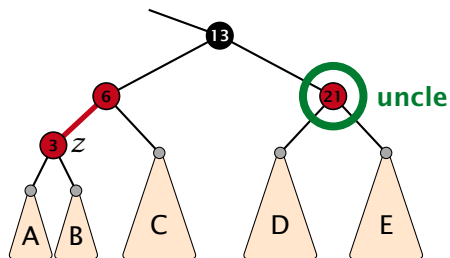
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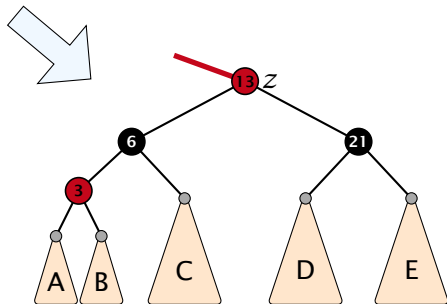
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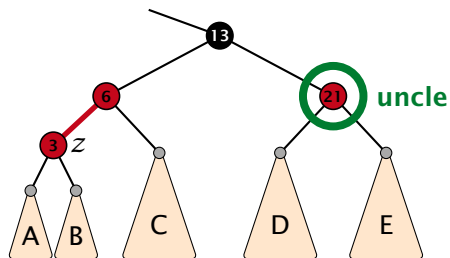
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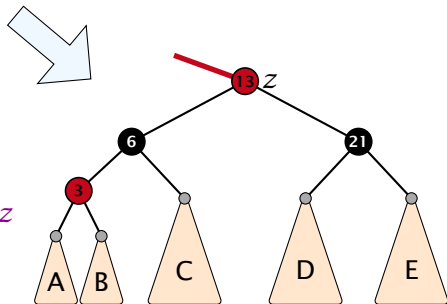
1. recolour
2. move  $z$  to grand-parent



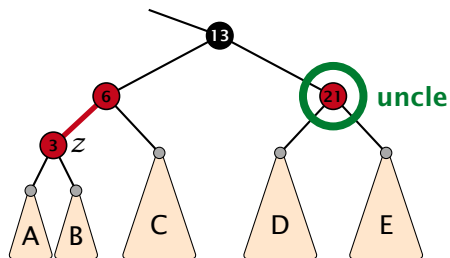
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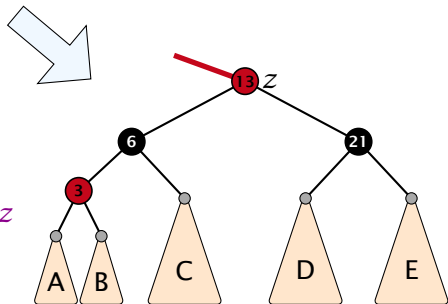
1. recolour
2. move  $z$  to grand-parent
3. invariant is fulfilled for new  $z$



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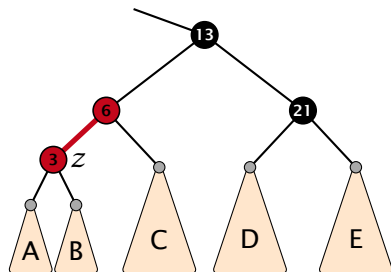


1. recolour
2. move *z* to grand-parent
3. invariant is fulfilled for new *z*
4. you made progress



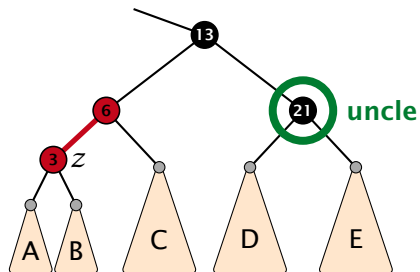
## Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree



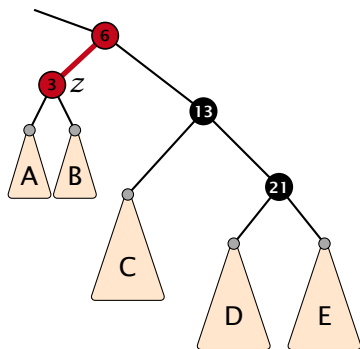
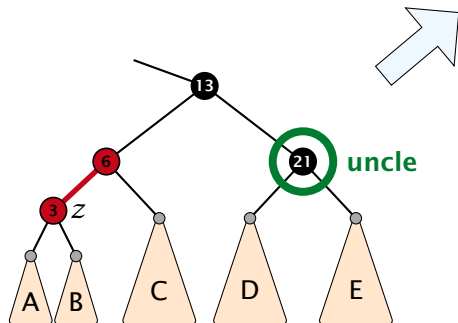
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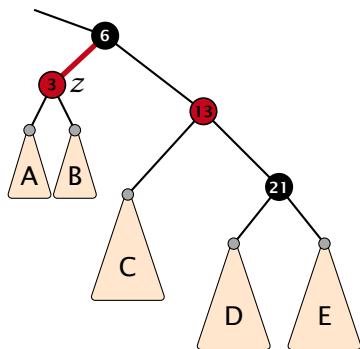
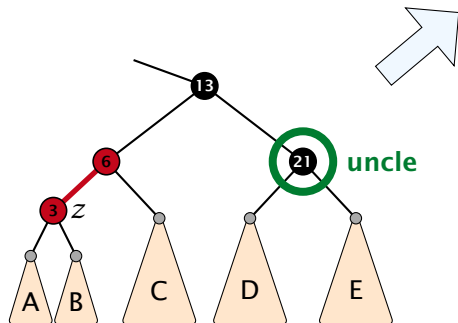
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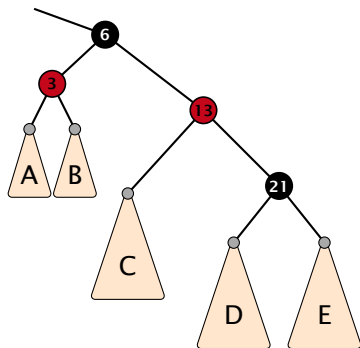
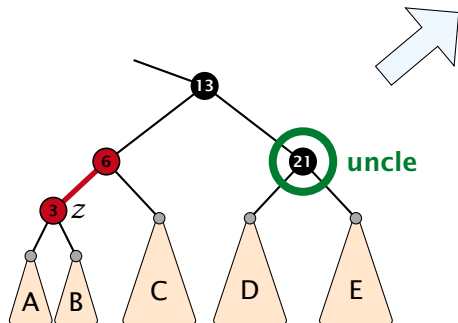
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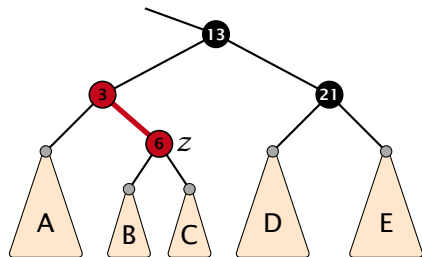
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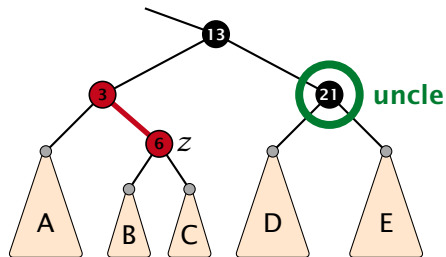
## Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards
3. you have Case 2b.



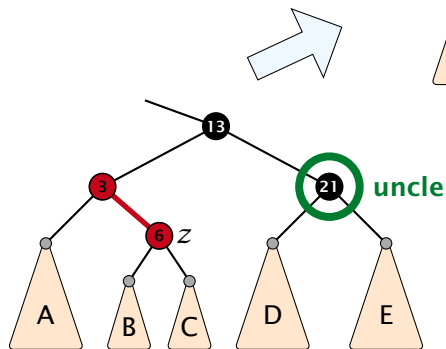
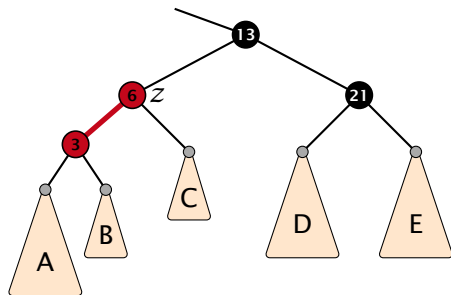
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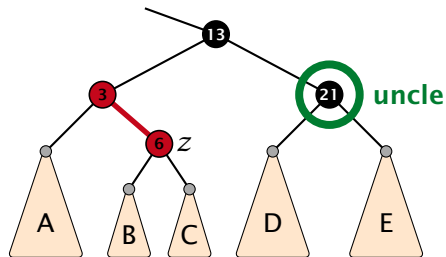
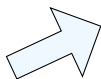
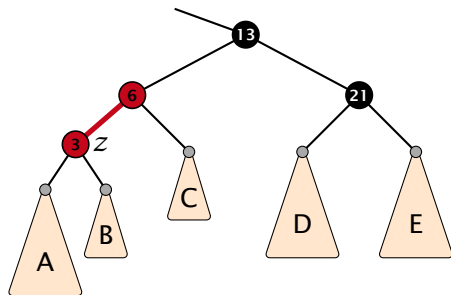
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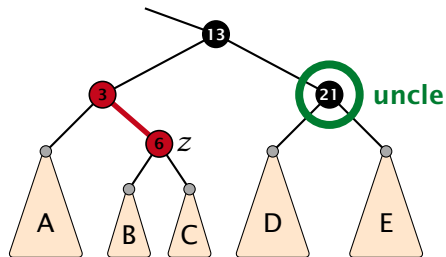
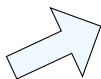
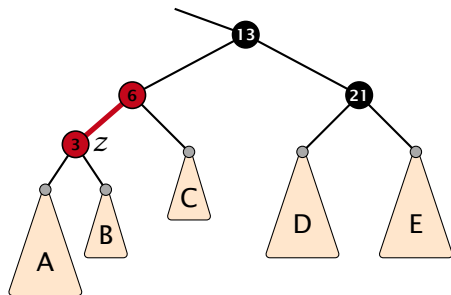
## Case 2a: Black uncle and z is right child

1. rotate around parent
2. move  $z$  downwards
3. you have Case 2b.



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# Red Black Trees: Insert

## Running time:

- ▶ Only Case 1 may repeat; but only  $h/2$  many steps, where  $h$  is the height of the tree.
- ▶ Case 2a → Case 2b → red-black tree
- ▶ Case 2b → red-black tree

Performing Case 1 at most  $\mathcal{O}(\log n)$  times and every other case at most once, we get a red-black tree. Hence  $\mathcal{O}(\log n)$  re-colorings and at most 2 rotations.



# Red Black Trees: Insert

## Running time:

- ▶ Only Case 1 may repeat; but only  $h/2$  many steps, where  $h$  is the height of the tree.
- ▶ Case 2a → Case 2b → red-black tree
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- ▶ Only Case 1 may repeat; but only  $h/2$  many steps, where  $h$  is the height of the tree.
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# Red Black Trees: Delete

First do a standard delete.

If the spliced out node  $x$  was red everything is fine.

If it was black there may be the following problems.

1. Parent and child of  $x$  were red; two adjacent red vertices.

2. If you delete the root, the root may now be red.

3. Every path from an ancestor of  $x$  to a descendant leaf of  $x$  changes the number of black nodes. Black height property might be violated.

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If the spliced out node  $x$  was red everything is fine.

If it was black there may be the following problems.

• Parent and child of  $x$  were red; two adjacent red nodes.

•  $x$  was the root; the root may now be red.

•  $x$  was left child of a red node; left-right imbalance.

•  $x$  was right child of a black node; Black Height property

might be violated.

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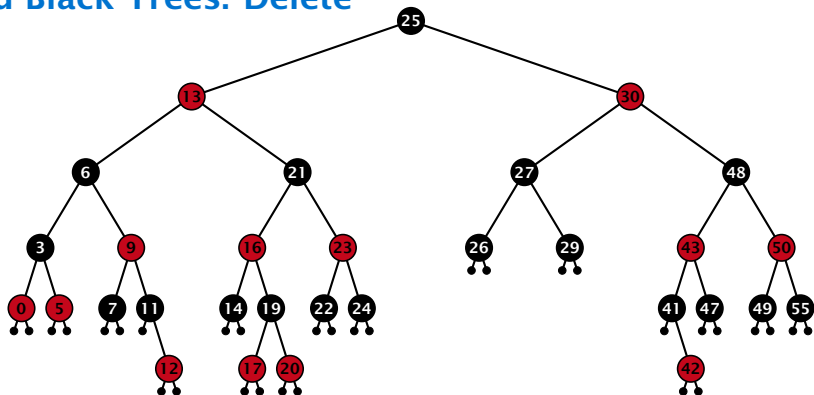
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If the spliced out node  $x$  was red everything is fine.

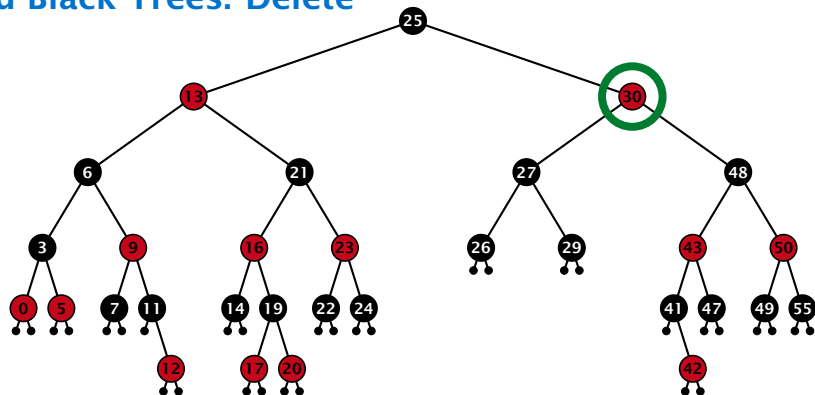
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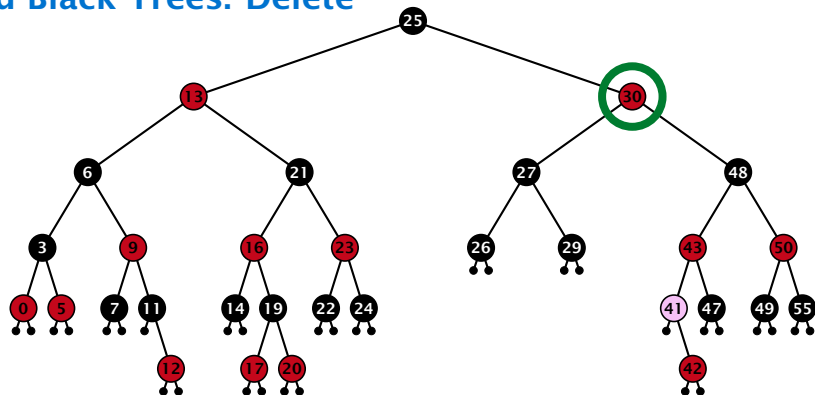


### Case 3:

Element has two children

- ▶ do normal delete
- ▶ when replacing content by content of successor, don't change color of node

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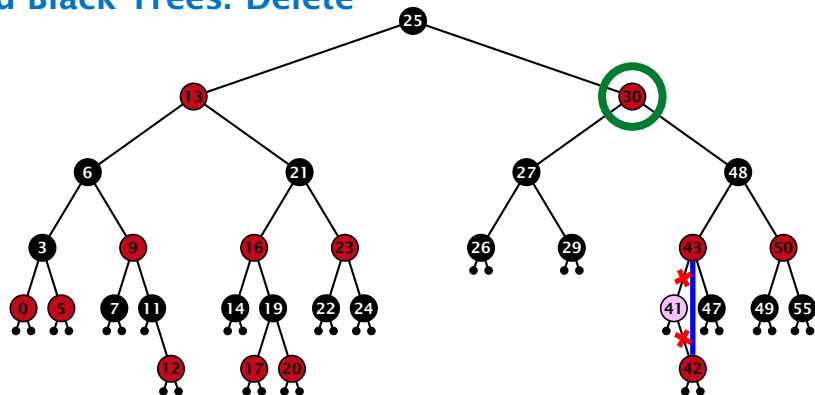


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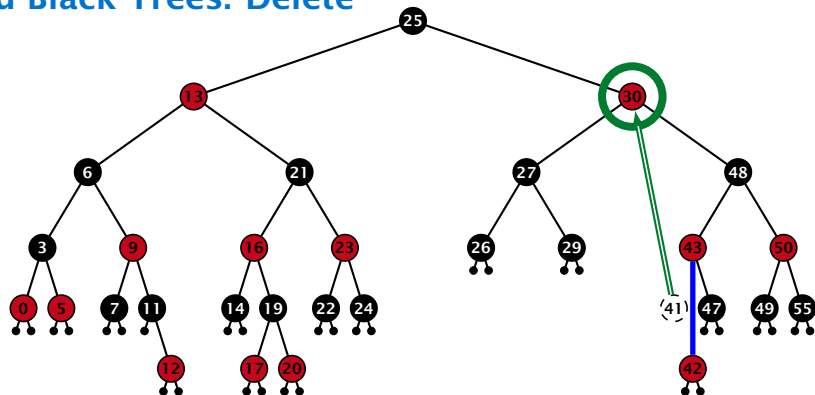


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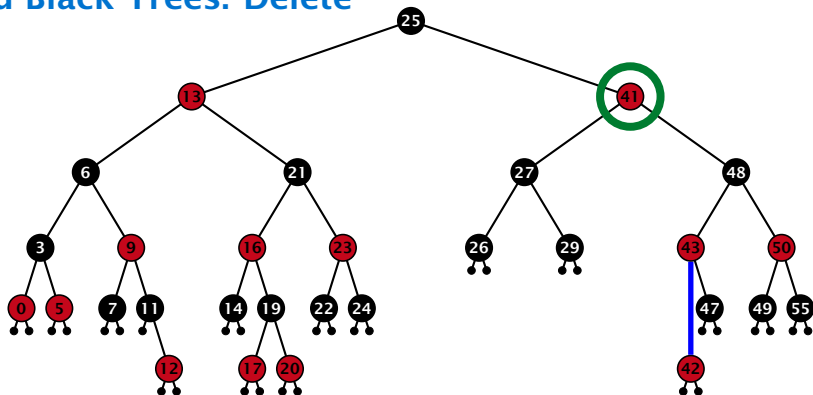


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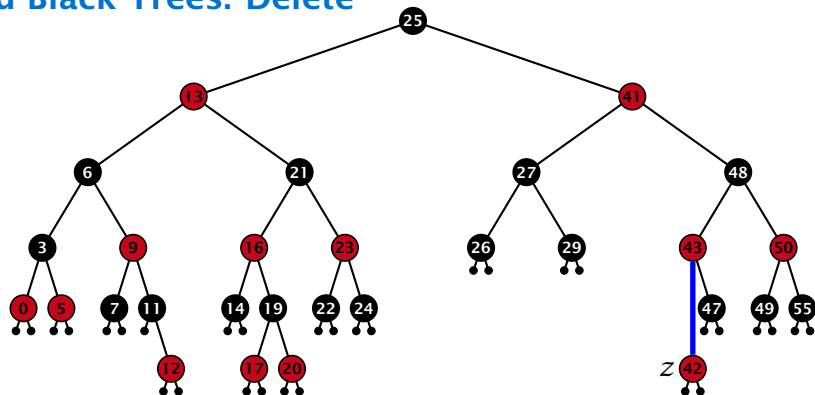
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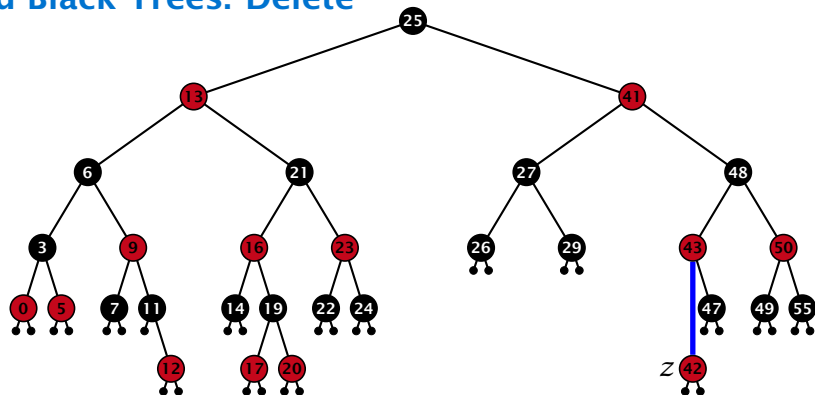
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### Delete:

- ▶ deleting black node messes up black-height property
- ▶ if  $z$  is red, we can simply color it black and everything is fine
- ▶ the problem is if  $z$  is black (e.g. a dummy-leaf); we call a fix-up procedure to fix the problem.

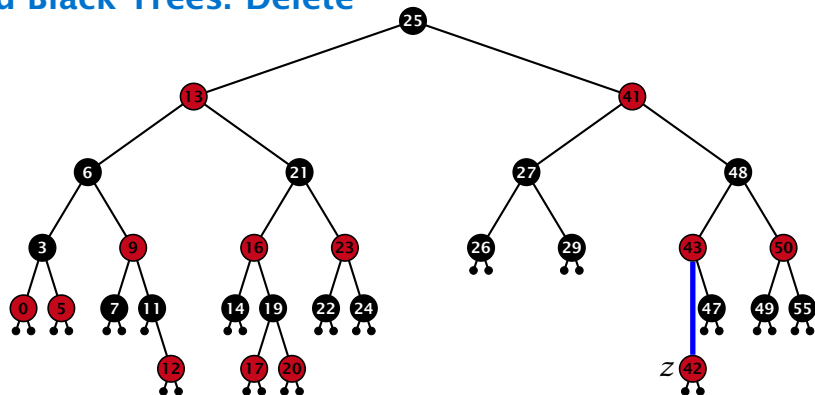
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## Invariant of the fix-up algorithm

- ▶ the node  $z$  is black
- ▶ if we “assign” a fake black unit to the edge from  $z$  to its parent then the black-height property is fulfilled

**Goal:** make rotations in such a way that you at some point can remove the fake black unit from the edge.

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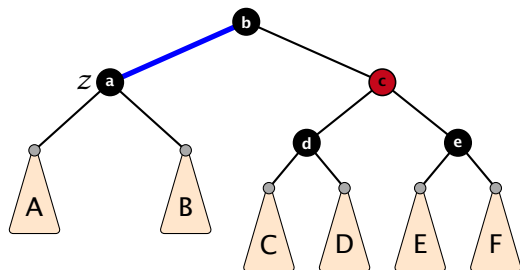
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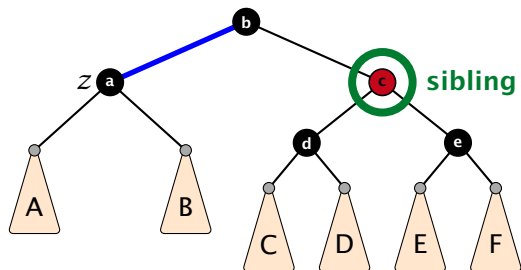
## Case 1: Sibling of $z$ is red



1. left-rotate around parent of  $z$
2. recolor nodes  $b$  and  $c$
3. the new sibling is black  
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4. Case 2 (special),  
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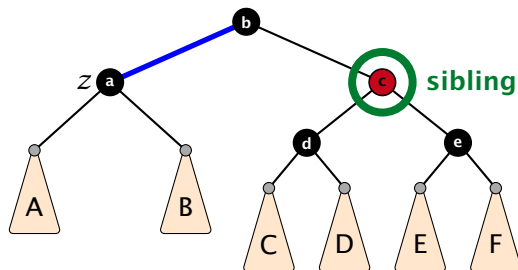


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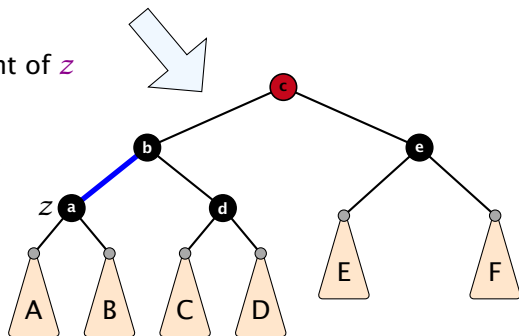


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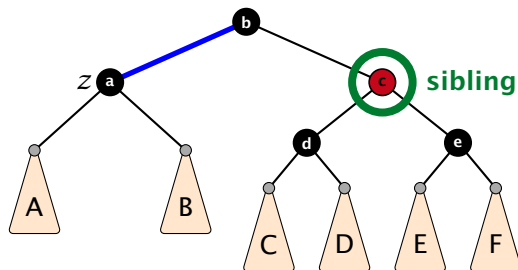
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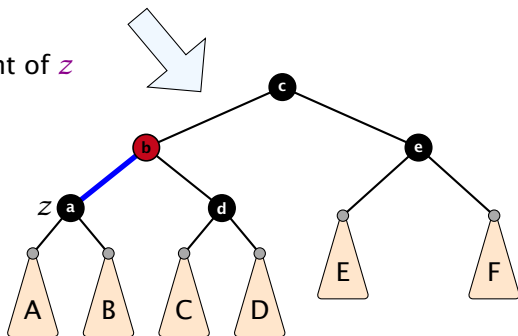
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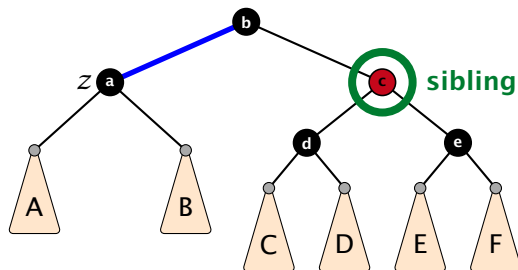
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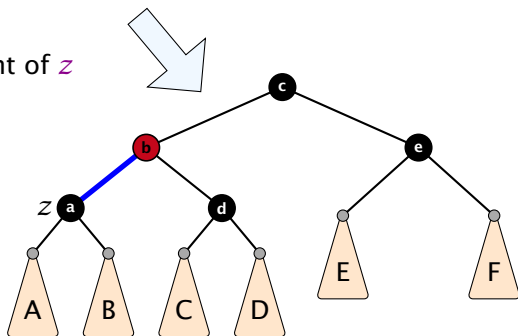
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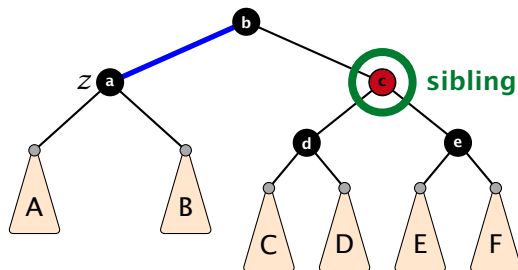
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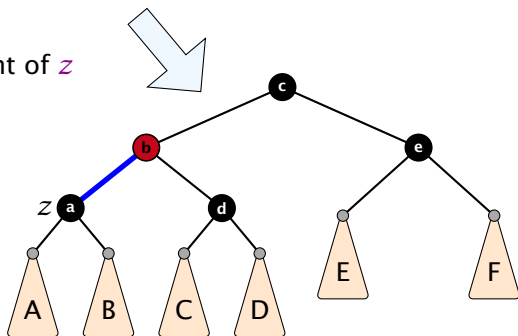
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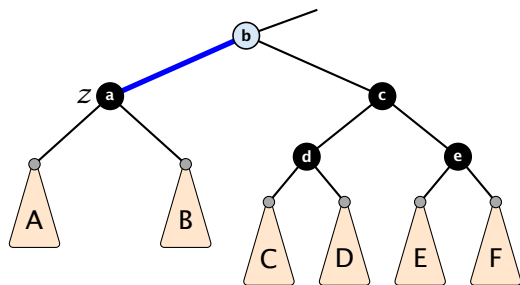
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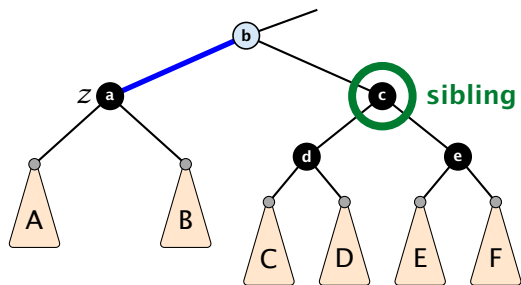
## Case 2: Sibling is black with two black children



1. re-color node  $c$
2. move fake black unit upwards
3. move  $z$  upwards
4. we made progress
5. if  $b$  is red we color it black and are done



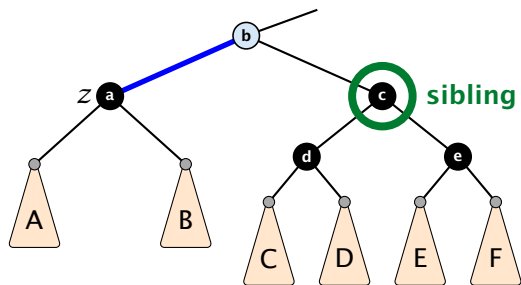
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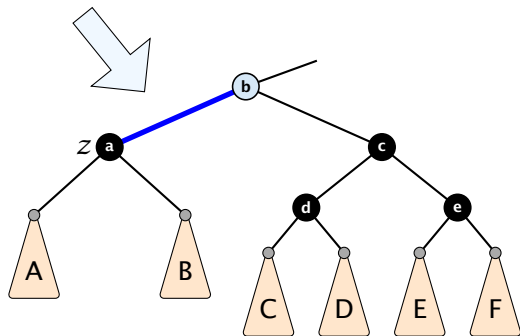
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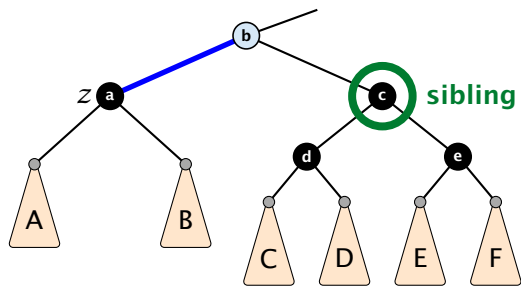
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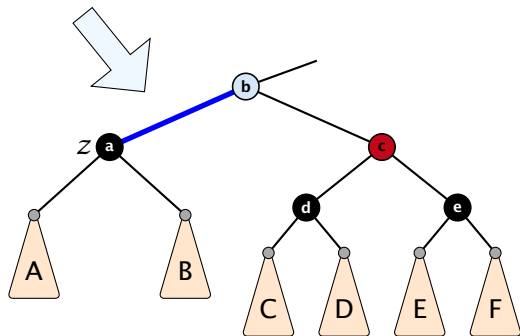
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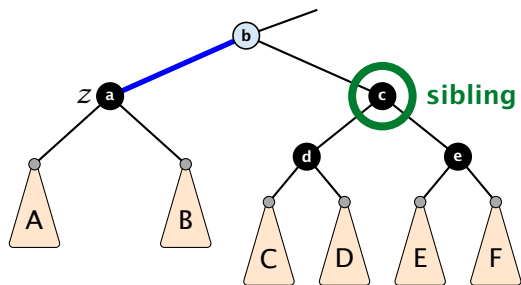


1. re-color node **c**
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5. if **b** is red we color it black and are done

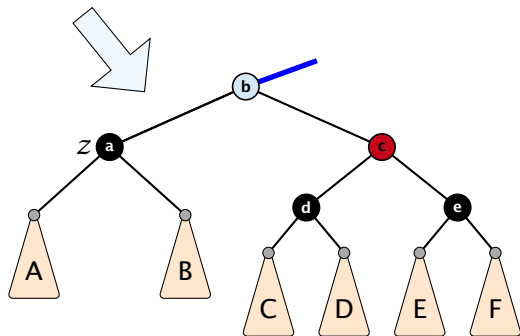




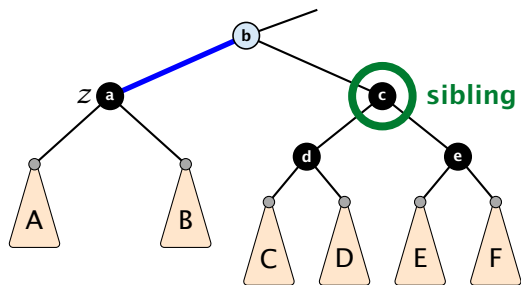
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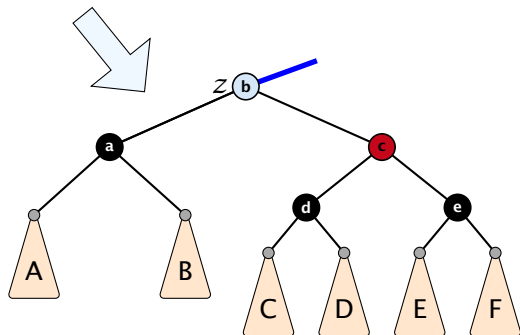
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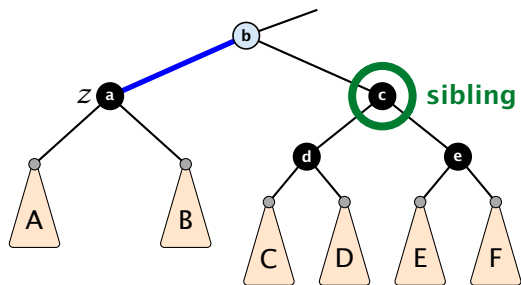
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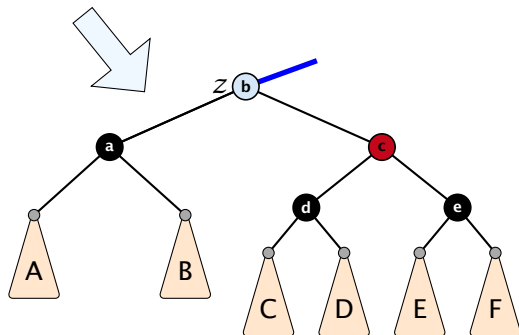
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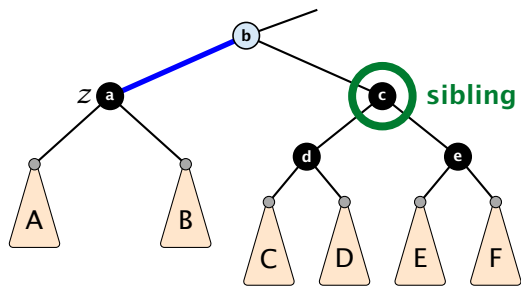
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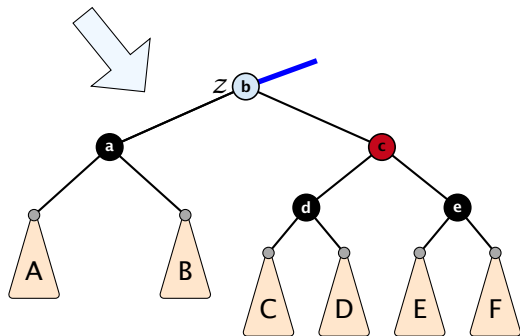
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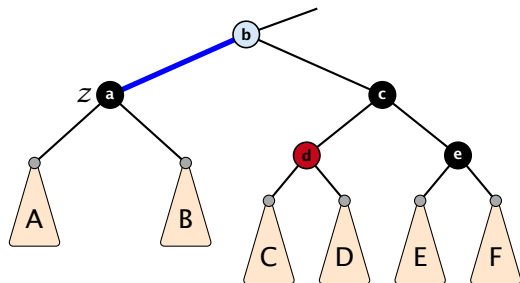


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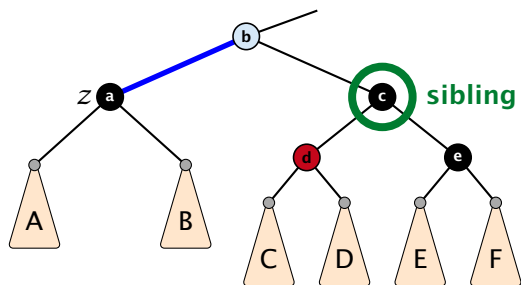
## Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor  $c$  and  $d$
3. new sibling is black with red right child (Case 4)



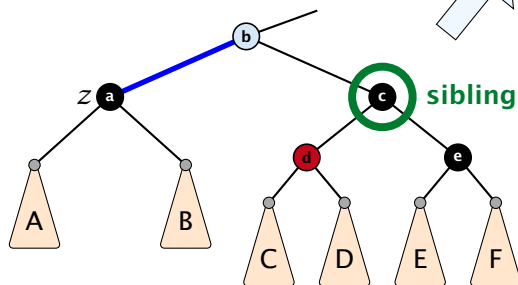
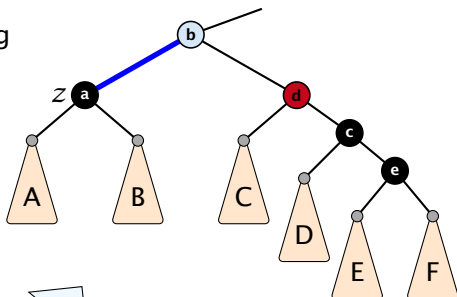
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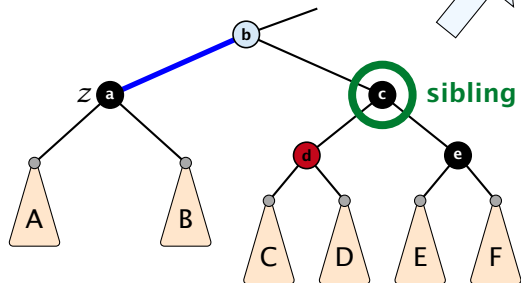
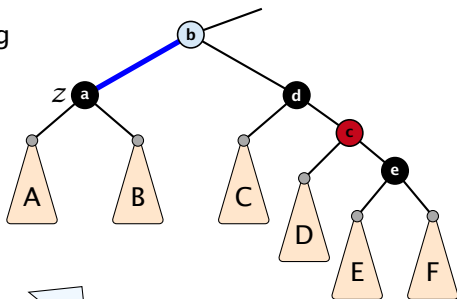
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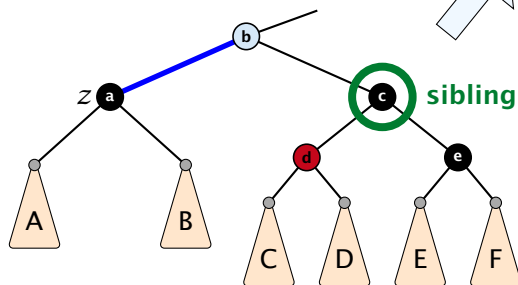
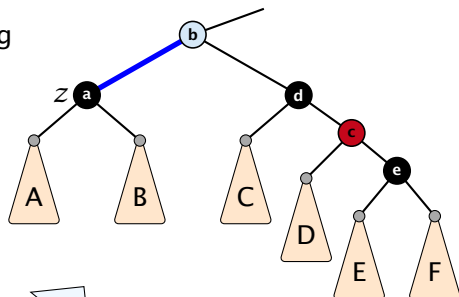
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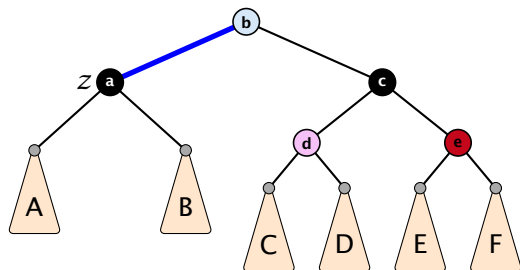


## Case 3: Sibling black with one black child to the right

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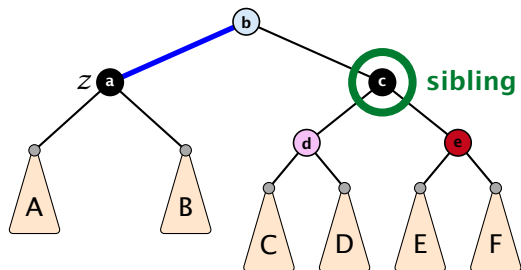
## Case 4: Sibling is black with red right child



1. left-rotate around  $b$
2. remove the fake black unit
3. recolor nodes  $b$ ,  $c$ , and  $e$
4. you have a valid red black tree



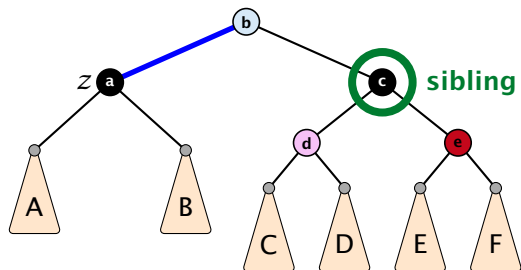
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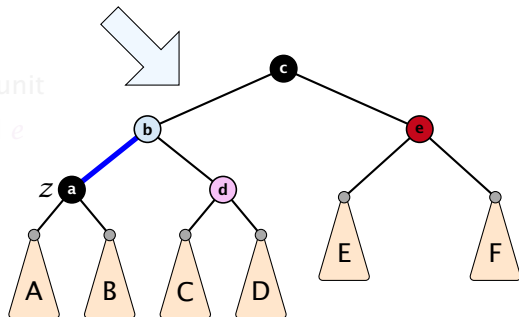
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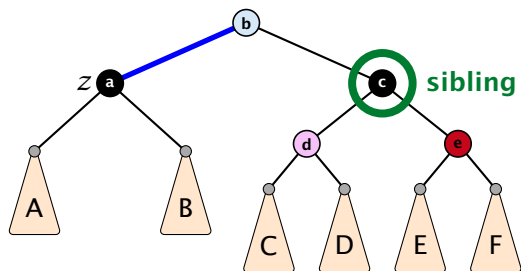
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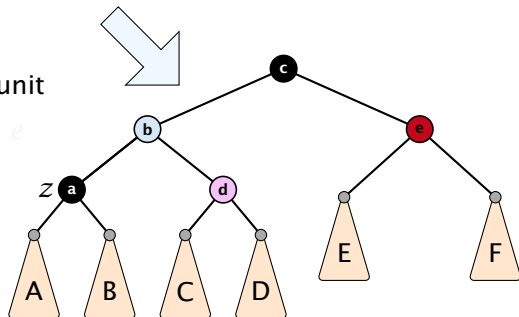
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2. remove the fake black unit
3. recolor nodes **b**, **c**, and **e**
4. you have a valid red black tree



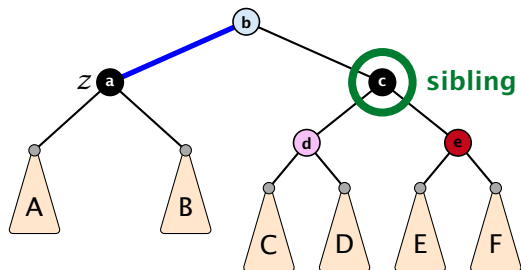
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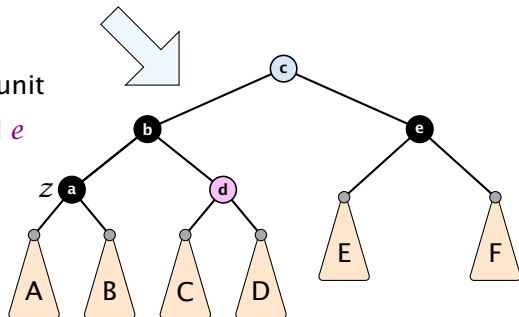
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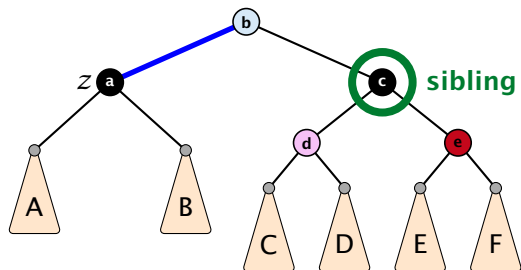
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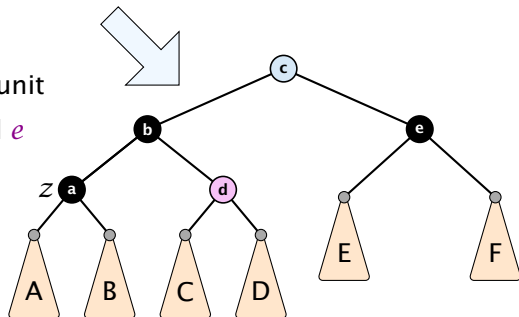
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# Splay Trees

## Disadvantage of balanced search trees:

- worst case; no advantage for easy inputs
- additional memory required
- complicated implementation

## Splay Trees:

- after access, an element is moved to the root (splay)
- repeated accesses are faster
- only amortized guarantee
- read-operation change the tree

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- amortized  $O(\log n)$  queries
- insert/delete may change the tree

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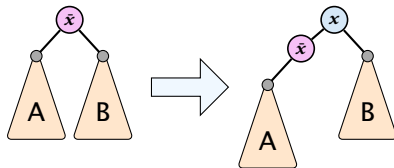
## **find( $x$ )**

- ▶ search for  $x$  according to a search tree
- ▶ let  $\tilde{x}$  be last element on search-path
- ▶ splay( $\tilde{x}$ )

# Splay Trees

## insert( $x$ )

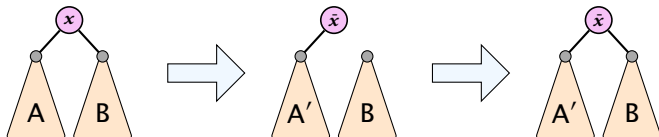
- ▶ search for  $x$ ;  $\bar{x}$  is last visited element during search (successor or predecessor of  $x$ )
- ▶ splay( $\bar{x}$ ) moves  $\bar{x}$  to the root
- ▶ insert  $x$  as new root



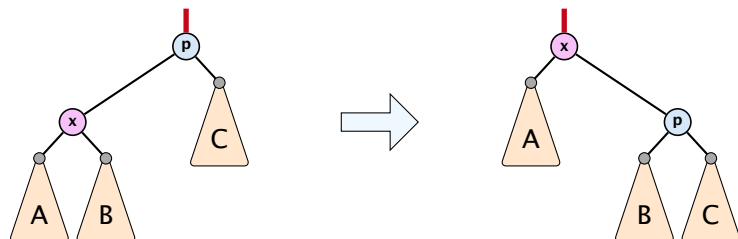
# Splay Trees

## delete( $x$ )

- ▶ search for  $x$ ; splay( $x$ ); remove  $x$
- ▶ search largest element  $\bar{x}$  in  $A$
- ▶ splay( $\bar{x}$ ) (on subtree  $A$ )
- ▶ connect root of  $B$  as right child of  $\bar{x}$



# Move to Root

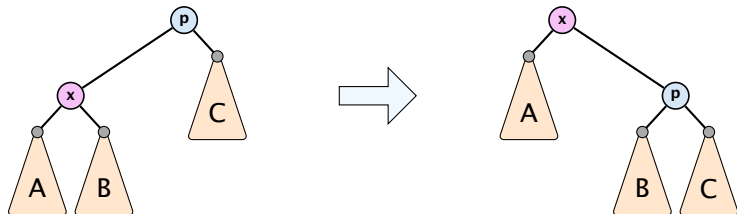


## How to bring element to root?

- ▶ one (bad) option: `moveToRoot(x)`
- ▶ iteratively do rotation around parent of  $x$  until  $x$  is root
- ▶ if  $x$  is left child do right rotation otw. left rotation



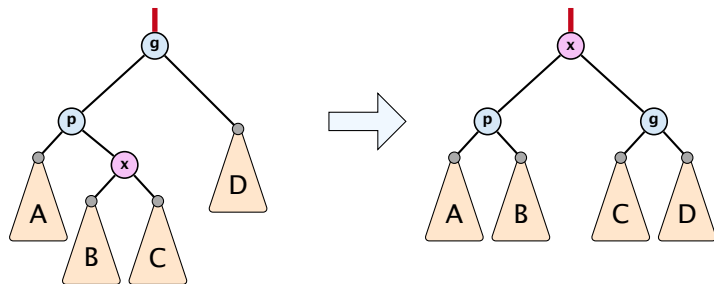
## Splay: Zig Case



**better option splay( $x$ ):**

- ▶ zig case: if  $x$  is child of root do left rotation or right rotation around parent

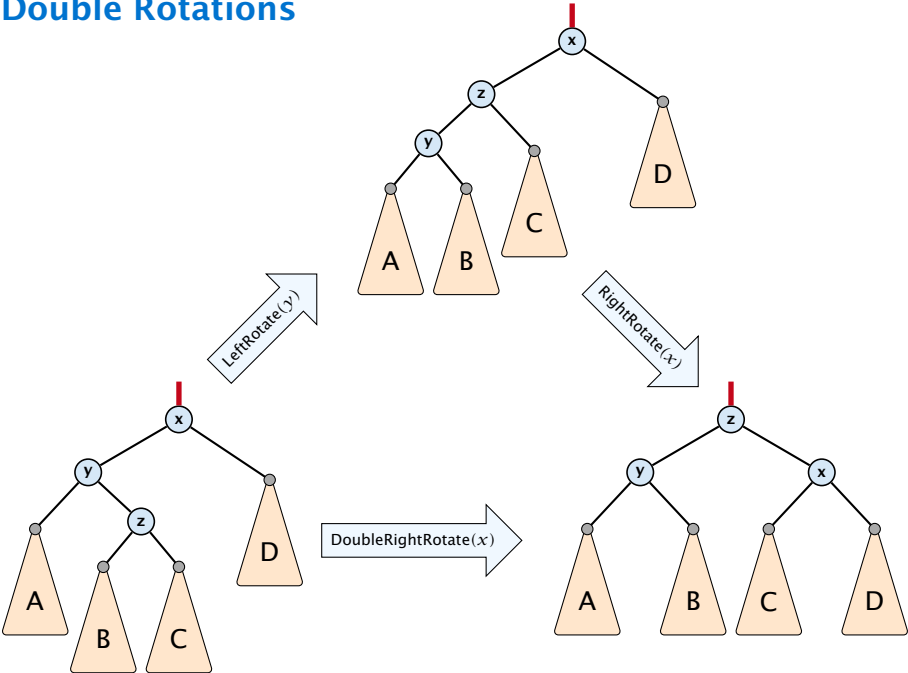
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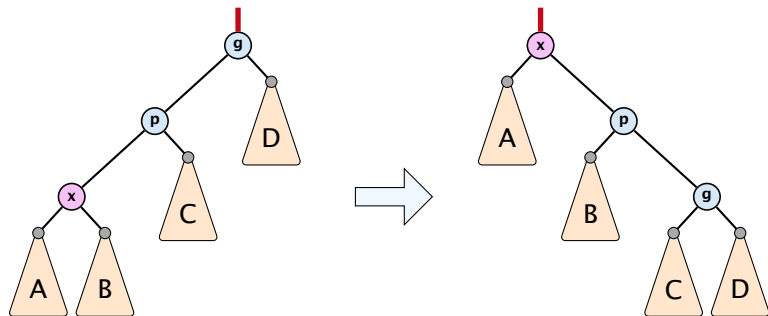
### better option $\text{splay}(x)$ :

- ▶ zigzag case: if  $x$  is right child and parent of  $x$  is left child (or  $x$  left child parent of  $x$  right child)
- ▶ do double right rotation around grand-parent (resp. double left rotation)

# Double Rotations



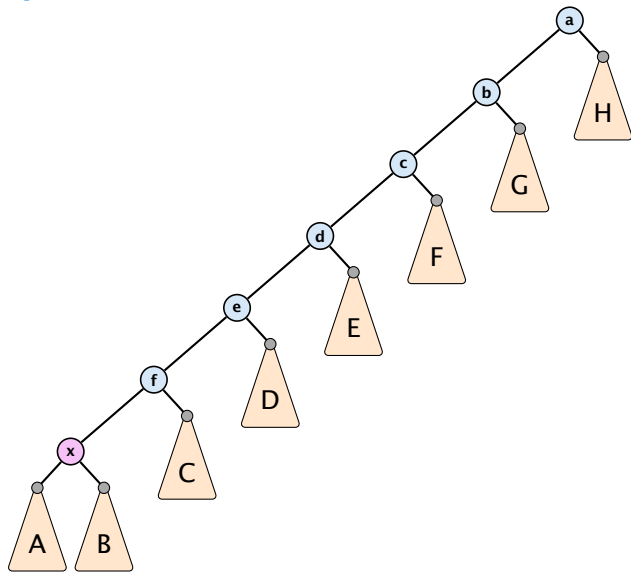
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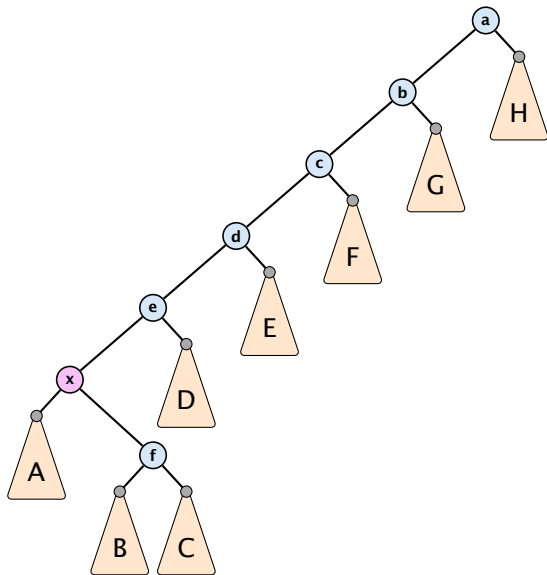
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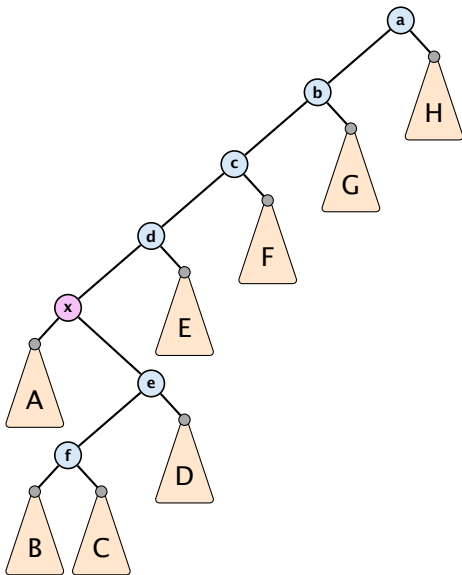
# Splay vs. Move to Root



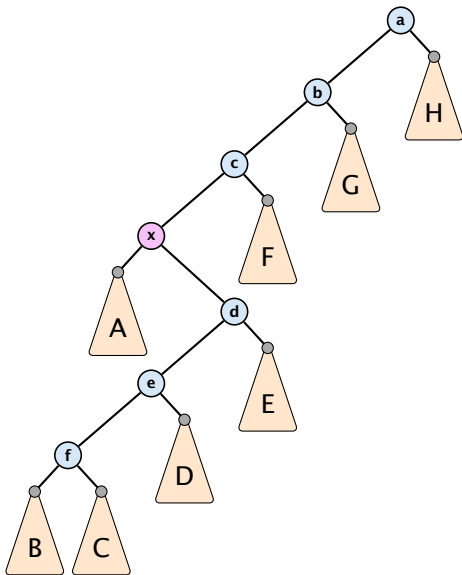
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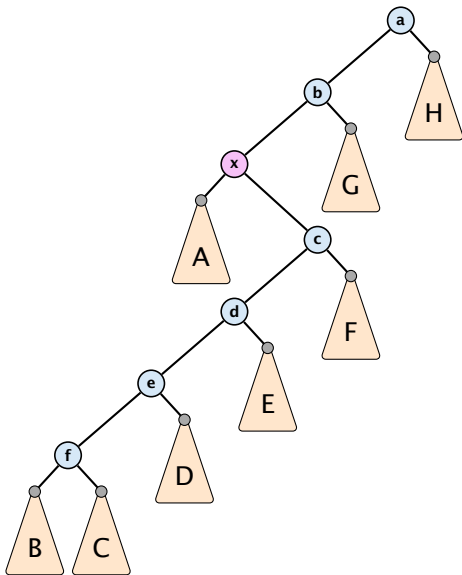


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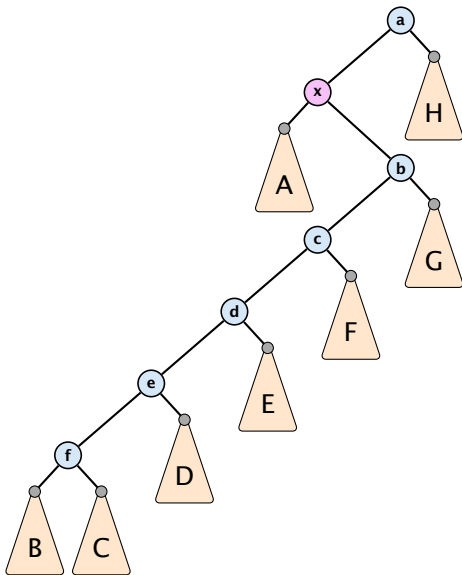




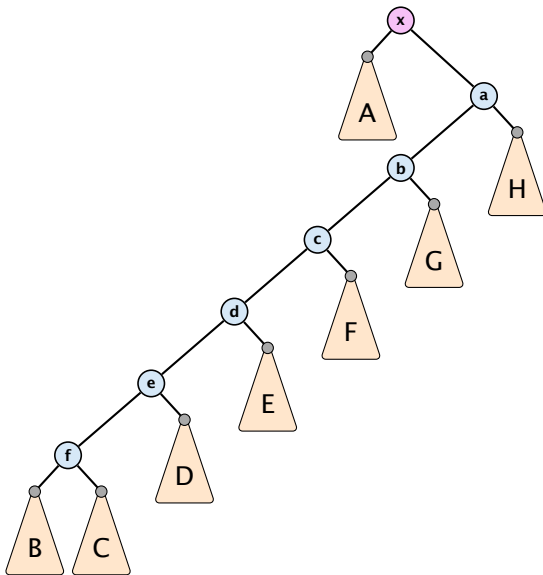
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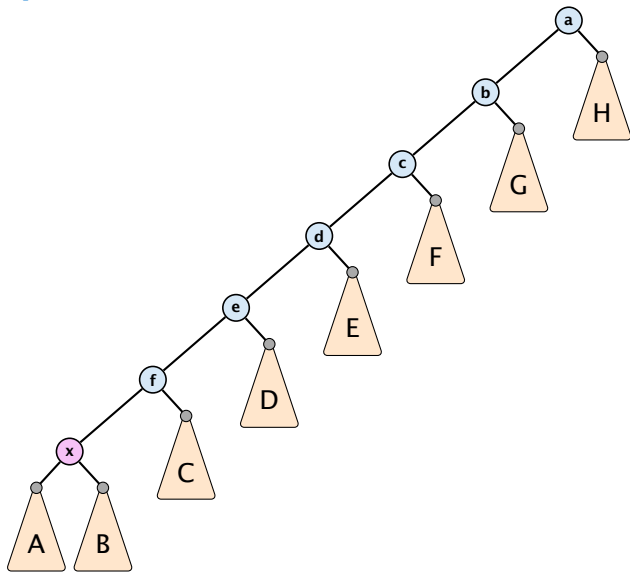
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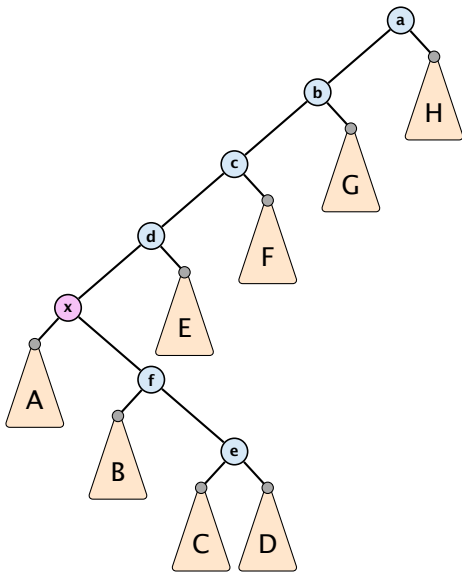
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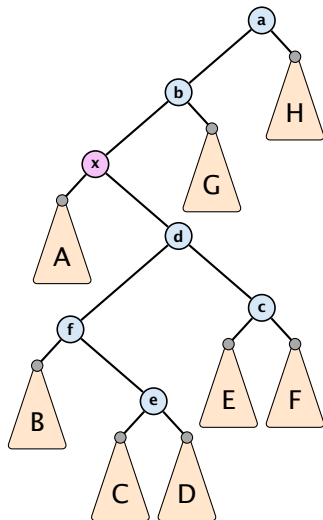
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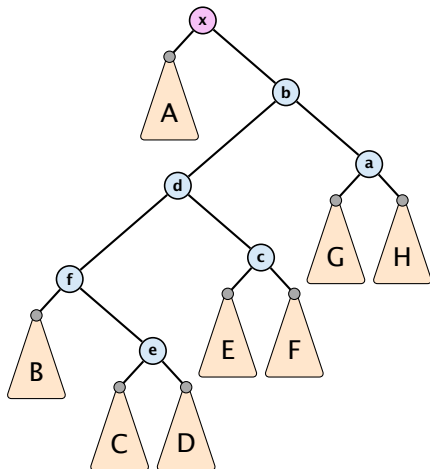
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# Static Optimality

Suppose we have a sequence of  $m$  find-operations.  $\text{find}(x)$  appears  $h_x$  times in this sequence.

The cost of a **static** search tree  $T$  is:

$$\text{cost}(T) = m + \sum_x h_x \text{depth}_T(x)$$

The total cost for processing the sequence on a splay-tree is  $\mathcal{O}(\text{cost}(T_{\min}))$ , where  $T_{\min}$  is an **optimal static search tree**.



# Dynamic Optimality

Let  $S$  be a sequence with  $m$  find-operations.

Let  $A$  be a data-structure based on a search tree:

- ▶ the cost for accessing element  $x$  is  $1 + \text{depth}(x)$ ;
- ▶ after accessing  $x$  the tree may be re-arranged through rotations;

## Conjecture:

A splay tree that only contains elements from  $S$  has cost  $\mathcal{O}(\text{cost}(A, S))$ , for processing  $S$ .

## Lemma 5

*Splay Trees have an **amortized** running time of  $\mathcal{O}(\log n)$  for all operations.*

# Amortized Analysis

## Definition 6

A data structure with operations  $\text{op}_1(), \dots, \text{op}_k()$  has amortized running times  $t_1, \dots, t_k$  for these operations if the following holds.

Suppose you are given a sequence of operations (**starting with an empty data-structure**) that operate on at most  $n$  elements, and let  $k_i$  denote the number of occurrences of  $\text{op}_i()$  within this sequence. Then the actual running time must be at most  $\sum_i k_i \cdot t_i(n)$ .

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$$\sum_{i=1}^k c_i \leq \sum_{i=1}^k c_i + \Phi(D_k) - \Phi(D_0) = \sum_{i=1}^k \hat{c}_i$$

This means the amortized costs can be used to derive a bound on the total cost.

# Example: Stack

## Stack

- ▶  $S.$  push()
- ▶  $S.$  pop()
- ▶  $S.$  multipop( $k$ ): removes  $k$  items from the stack. If the stack currently contains less than  $k$  items it empties the stack.
- ▶ The user has to ensure that pop and multipop do not generate an underflow.

## Actual cost:

- ▶  $S.$  push(): cost 1.
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### Incrementing a binary counter:

Consider a computational model where each bit-operation costs one time-unit.

Incrementing an  $n$ -bit binary counter may require to examine  $n$ -bits, and maybe change them.

### Actual cost:

- ▶ Changing bit from 0 to 1: cost 1.
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Choose potential function  $\Phi(x) = k$ , where  $k$  denotes the number of ones in the binary representation of  $x$ .

**Amortized cost:**

Let  $z$  denote the number of consecutive ones in the least significant bit positions. An increment involves  $z$  operations, and one  $0 \rightarrow 1$  operation.

Hence, the amortized cost is

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- ▶ **Increment:** Let  $k$  denotes the number of consecutive ones in the least significant bit-positions. An increment involves  $k$  (1  $\rightarrow$  0)-operations, and one (0  $\rightarrow$  1)-operation.

Hence, the amortized cost is  $k\hat{C}_{1 \rightarrow 0} + \hat{C}_{0 \rightarrow 1} \leq 2$ .

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Hence, the amortized cost is  $k\hat{C}_{1 \rightarrow 0} + \hat{C}_{0 \rightarrow 1} \leq 2$ .



# Splay Trees

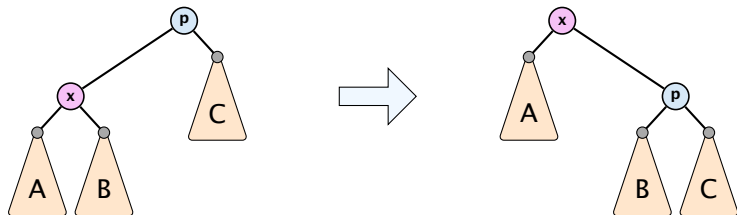
## potential function for splay trees:

- ▶ size  $s(x) = |T_x|$
- ▶ rank  $r(x) = \log_2(s(x))$
- ▶  $\Phi(T) = \sum_{v \in T} r(v)$

amortized cost = real cost + potential change

The cost is essentially the cost of the splay-operation, which is 1 plus the number of rotations.

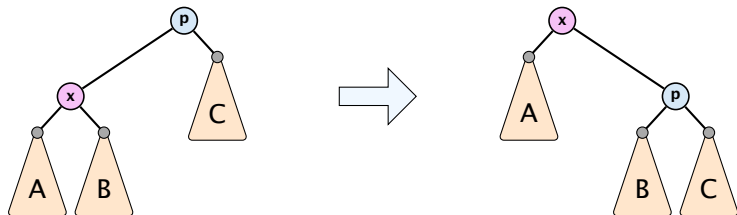
## Splay: Zig Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x) \\ &\leq r'(x) - r(x)\end{aligned}$$

$$\text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x))$$

## Splay: Zig Case



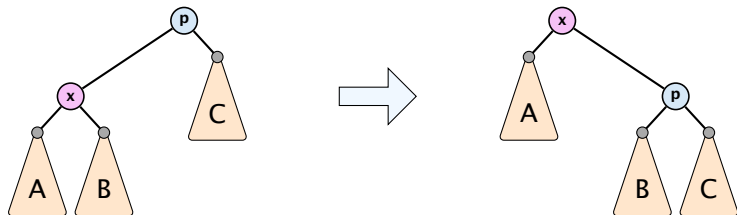
$$\Delta\Phi = r'(x) + r'(p) - r(x) - r(p)$$

$$= r'(p) - r(x)$$

$$\leq r'(x) - r(x)$$

$$\text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x))$$

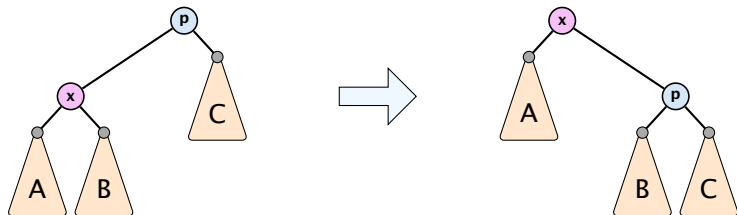
## Splay: Zig Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x) \\ &\leq r'(x) - r(x)\end{aligned}$$

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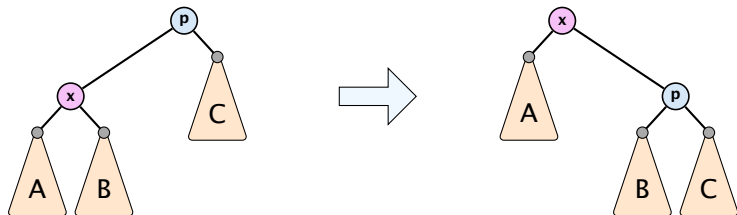
## Splay: Zig Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x) \\ &\leq r'(x) - r(x)\end{aligned}$$

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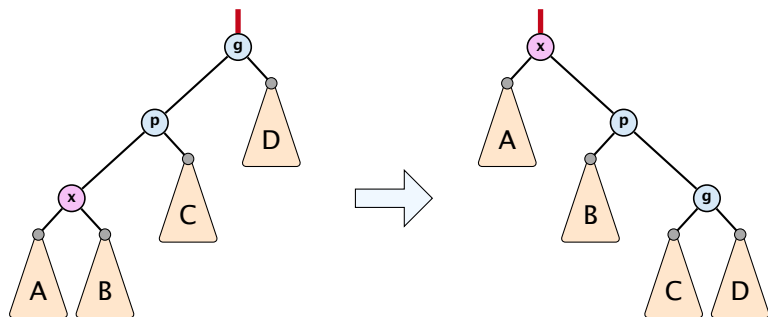
## Splay: Zig Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) - r(x) - r(p) \\ &= r'(p) - r(x) \\ &\leq r'(x) - r(x)\end{aligned}$$

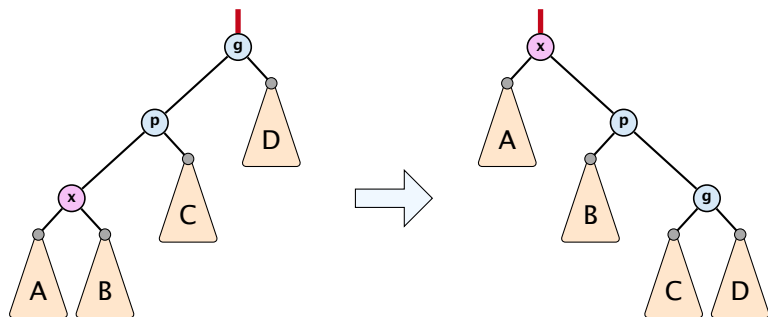
$$\text{cost}_{\text{zig}} \leq 1 + 3(r'(x) - r(x))$$

## Splay: Zigzig Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(x) + r'(g) - r(x) - r(x) \\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x) \\ &= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x)) \\ &\leq -2 + 3(r'(x) - r(x)) \Rightarrow \text{cost}_{\text{zigzig}} \leq 3(r'(x) - r(x))\end{aligned}$$

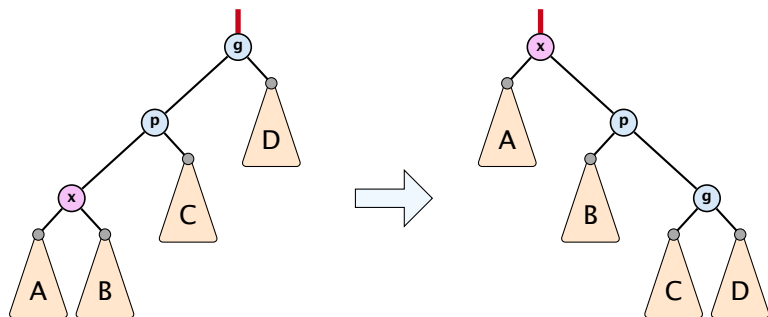
## Splay: Zigzig Case



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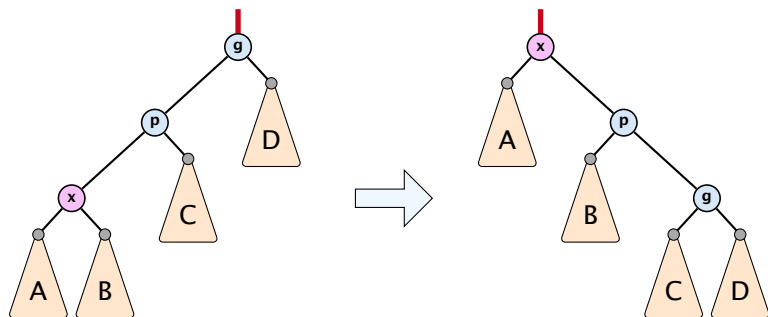


## Splay: Zigzig Case



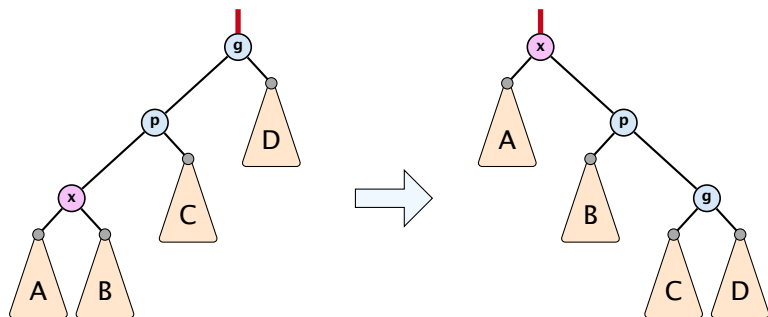
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## Splay: Zigzig Case



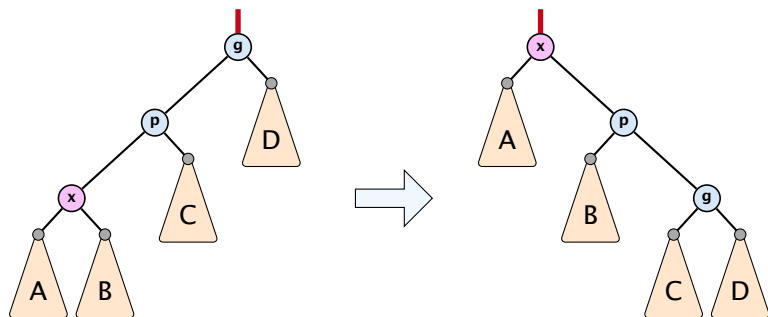
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## Splay: Zigzig Case



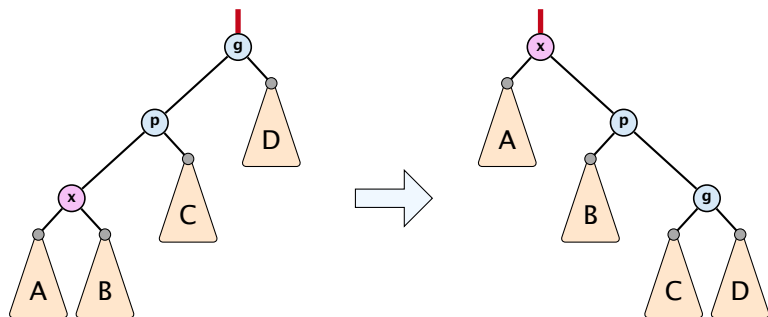
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## Splay: Zigzig Case



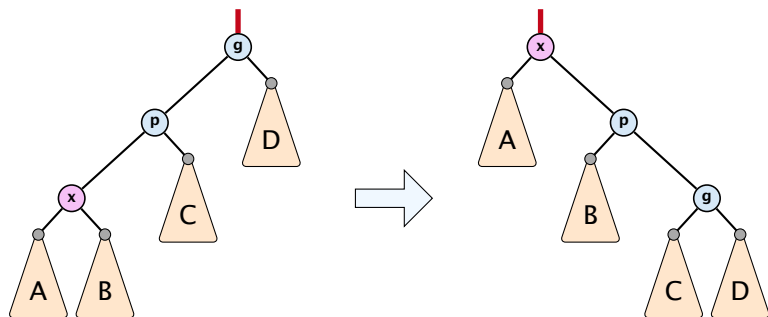
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## Splay: Zigzig Case



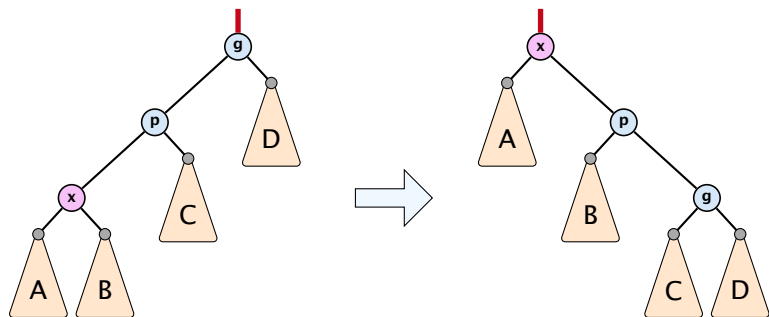
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(x) + r'(g) - r(x) - r(x) \\ &= r'(x) + r'(g) + r(x) - 3r'(x) + 3r'(x) - r(x) - 2r(x) \\ &= -2r'(x) + r'(g) + r(x) + 3(r'(x) - r(x)) \\ &\leq -2 + 3(r'(x) - r(x)) \quad \Rightarrow \text{cost}_{\text{zigzig}} \leq 3(r'(x) - r(x))\end{aligned}$$

## Splay: Zigzig Case



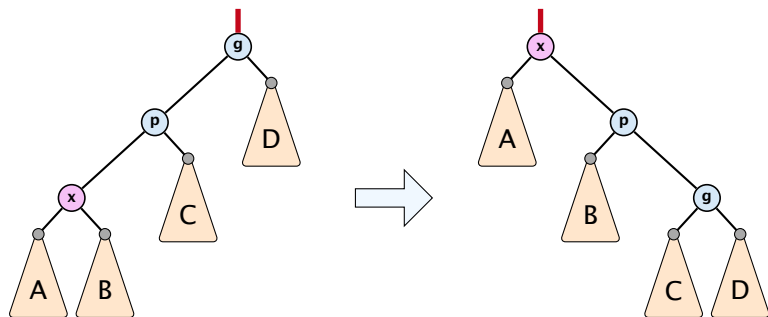
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## Splay: Zigzig Case



$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s(x)) + \log(s'(g)) - 2\log(s'(x))) \\ &= \frac{1}{2} \log\left(\frac{s(x)}{s'(x)}\right) + \frac{1}{2} \log\left(\frac{s'(g)}{s'(x)}\right) \\ &\leq \log\left(\frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)}\right) \leq \log\left(\frac{1}{2}\right) = -1 \end{aligned}$$

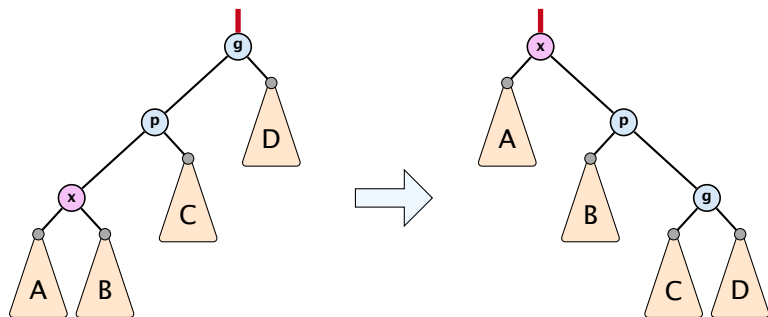
## Splay: Zigzig Case



$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s(x)) + \log(s'(g)) - 2\log(s'(x))) \\ &= \frac{1}{2} \log\left(\frac{s(x)}{s'(x)}\right) + \frac{1}{2} \log\left(\frac{s'(g)}{s'(x)}\right) \\ &\leq \log\left(\frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)}\right) \leq \log\left(\frac{1}{2}\right) = -1 \end{aligned}$$

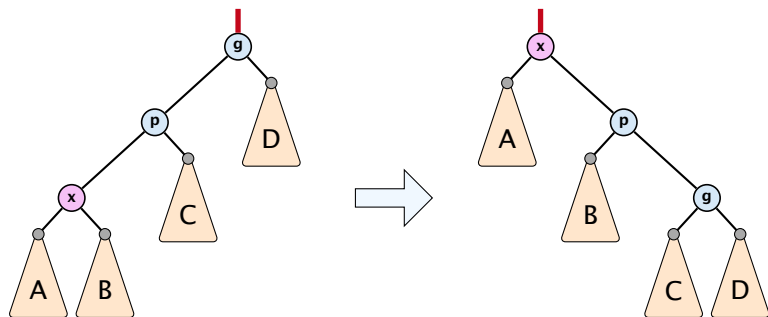


## Splay: Zigzig Case



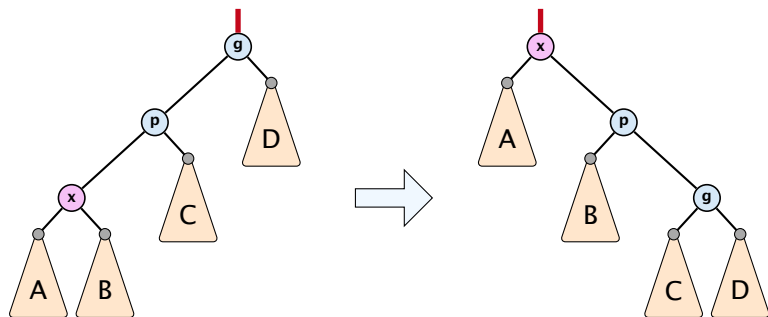
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \\ &= \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right) \\ &\leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1 \end{aligned}$$

## Splay: Zigzig Case



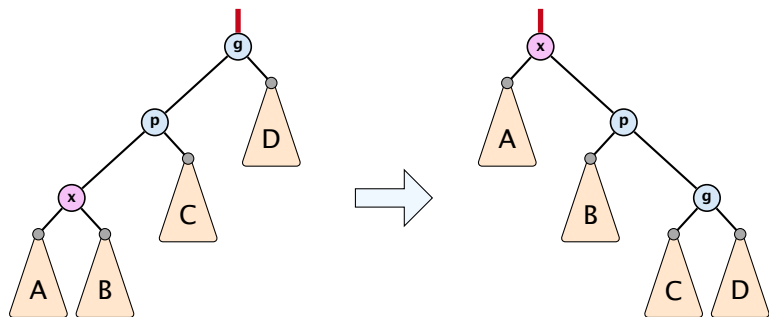
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \\ &= \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right) \\ &\leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1 \end{aligned}$$

## Splay: Zigzig Case



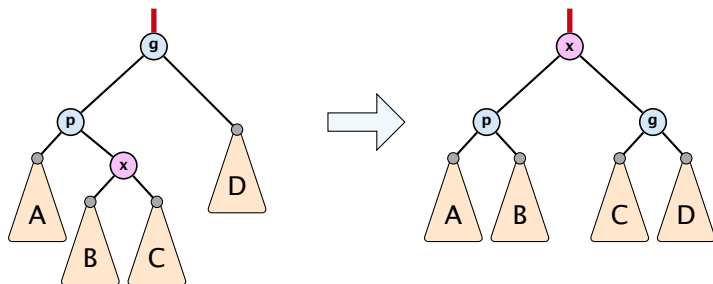
$$\begin{aligned} & \frac{1}{2} \left( r(x) + r'(g) - 2r'(x) \right) \\ &= \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \\ &= \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right) \\ &\leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1 \end{aligned}$$

## Splay: Zigzig Case



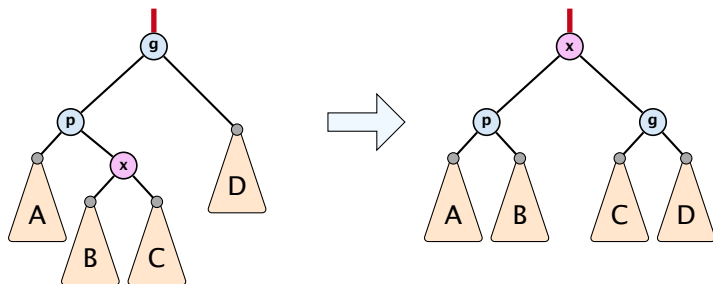
$$\begin{aligned} & \frac{1}{2} (r(x) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} \left( \log(s(x)) + \log(s'(g)) - 2 \log(s'(x)) \right) \\ &= \frac{1}{2} \log \left( \frac{s(x)}{s'(x)} \right) + \frac{1}{2} \log \left( \frac{s'(g)}{s'(x)} \right) \\ &\leq \log \left( \frac{1}{2} \frac{s(x)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)} \right) \leq \log \left( \frac{1}{2} \right) = -1 \end{aligned}$$

## Splay: Zigzag Case



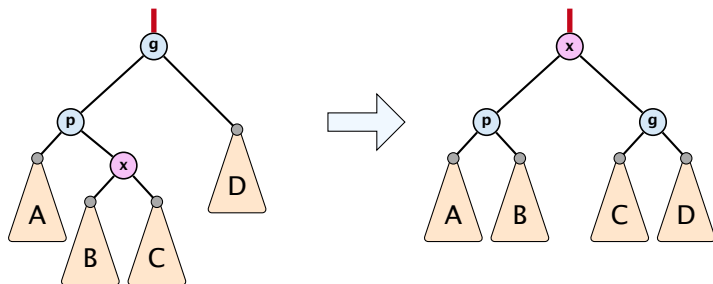
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(p) + r'(g) - r(x) - r(x) \\ &= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \\ &\leq -2 + 2(r'(x) - r(x)) \Rightarrow \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x))\end{aligned}$$

## Splay: Zigzag Case



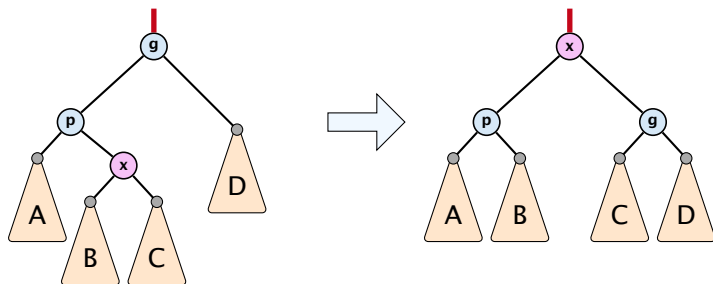
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(p) + r'(g) - r(x) - r(x) \\ &= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \\ &\leq -2 + 2(r'(x) - r(x)) \Rightarrow \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x))\end{aligned}$$

## Splay: Zigzag Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(p) + r'(g) - r(x) - r(x) \\ &= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \\ &\leq -2 + 2(r'(x) - r(x)) \Rightarrow \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x))\end{aligned}$$

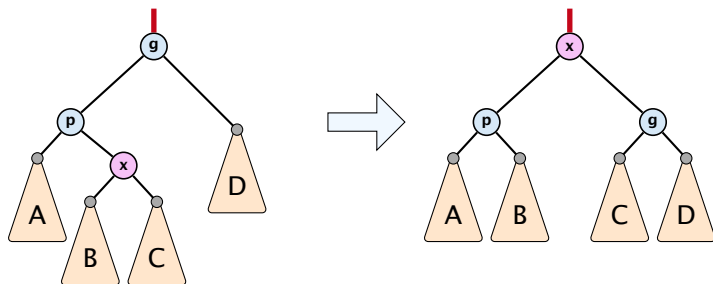
## Splay: Zigzag Case



$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(p) + r'(g) - r(x) - r(x) \\ &= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \\ &\leq -2 + 2(r'(x) - r(x)) \Rightarrow \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x))\end{aligned}$$

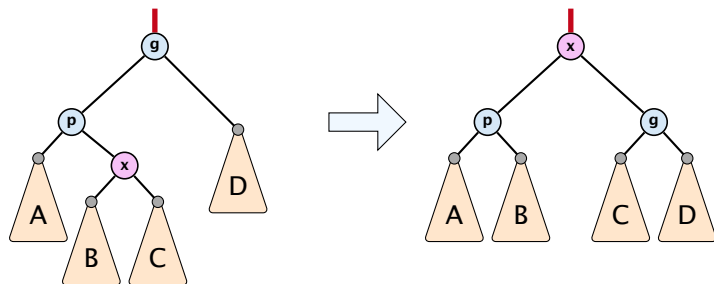


## Splay: Zigzag Case



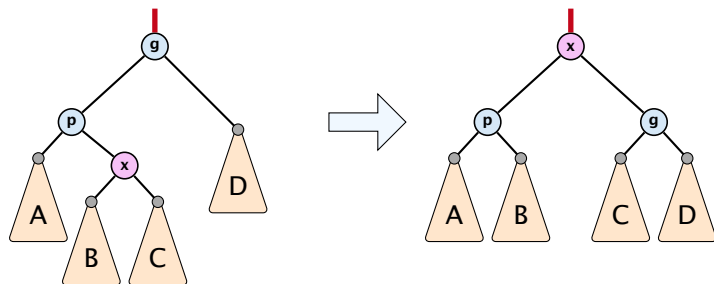
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(p) + r'(g) - r(x) - r(x) \\ &= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \\ &\leq -2 + 2(r'(x) - r(x)) \Rightarrow \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x))\end{aligned}$$

## Splay: Zigzag Case



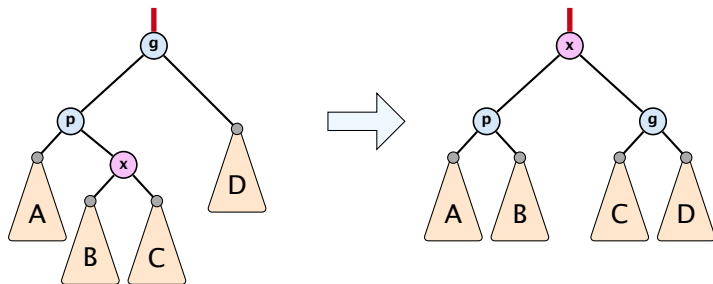
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(p) + r'(g) - r(x) - r(x) \\ &= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \\ &\leq -2 + 2(r'(x) - r(x)) \Rightarrow \text{cost}_{\text{zigzag}} \leq 3(r'(x) - r(x))\end{aligned}$$

## Splay: Zigzag Case



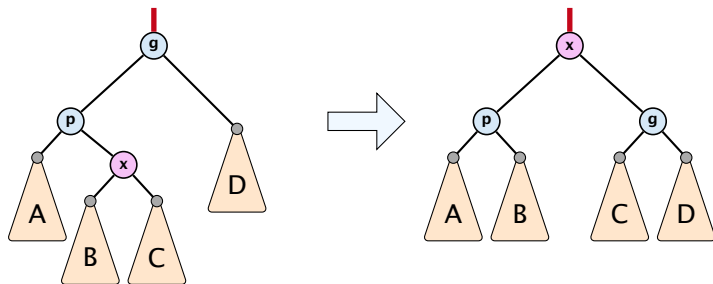
$$\begin{aligned}\Delta\Phi &= r'(x) + r'(p) + r'(g) - r(x) - r(p) - r(g) \\ &= r'(p) + r'(g) - r(x) - r(p) \\ &\leq r'(p) + r'(g) - r(x) - r(x) \\ &= r'(p) + r'(g) - 2r'(x) + 2r'(x) - 2r(x) \\ &\leq -2 + 2(r'(x) - r(x)) \Rightarrow \text{COST}_{\text{zigzag}} \leq 3(r'(x) - r(x))\end{aligned}$$

## Splay: Zigzag Case



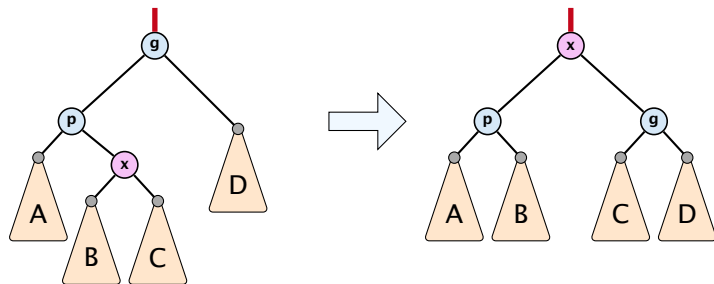
$$\begin{aligned} & \frac{1}{2} (r'(p) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s'(p)) + \log(s'(g)) - 2 \log(s'(x))) \\ &\leq \log\left(\frac{1}{2} \frac{s'(p)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)}\right) \leq \log\left(\frac{1}{2}\right) = -1 \end{aligned}$$

## Splay: Zigzag Case



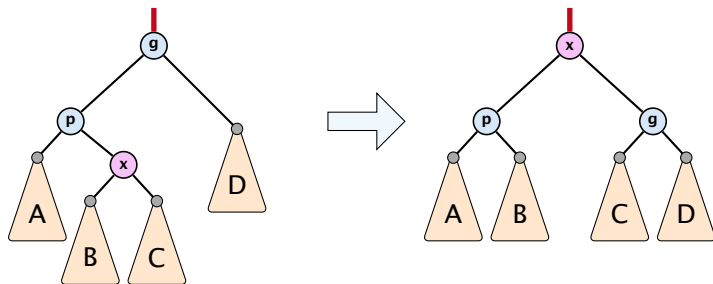
$$\begin{aligned} & \frac{1}{2} (r'(p) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))) \\ &\leq \log\left(\frac{1}{2} \frac{s'(p)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)}\right) \leq \log\left(\frac{1}{2}\right) = -1 \end{aligned}$$

## Splay: Zigzag Case



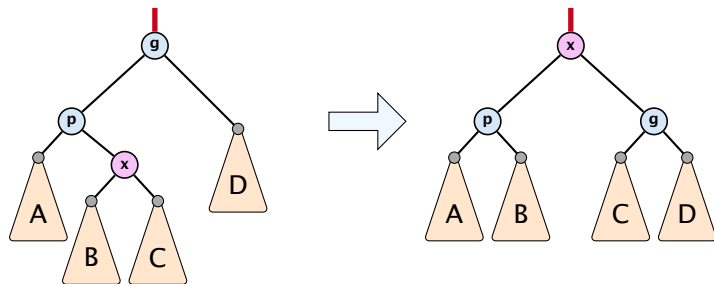
$$\begin{aligned} & \frac{1}{2} (r'(p) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))) \\ &\leq \log\left(\frac{1}{2} \frac{s'(p)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)}\right) \leq \log\left(\frac{1}{2}\right) = -1 \end{aligned}$$

## Splay: Zigzag Case



$$\begin{aligned} & \frac{1}{2} (r'(p) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))) \\ &\leq \log\left(\frac{1}{2} \frac{s'(p)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)}\right) \leq \log\left(\frac{1}{2}\right) = -1 \end{aligned}$$

## Splay: Zigzag Case



$$\begin{aligned} & \frac{1}{2} (r'(p) + r'(g) - 2r'(x)) \\ &= \frac{1}{2} (\log(s'(p)) + \log(s'(g)) - 2\log(s'(x))) \\ &\leq \log\left(\frac{1}{2} \frac{s'(p)}{s'(x)} + \frac{1}{2} \frac{s'(g)}{s'(x)}\right) \leq \log\left(\frac{1}{2}\right) = -1 \end{aligned}$$



Amortized cost of the whole splay operation:

$$\begin{aligned} &\leq 1 + 1 + \sum_{\text{steps } t} 3(r_t(x) - r_{t-1}(x)) \\ &= 2 + r(\text{root}) - r_0(x) \\ &\leq \mathcal{O}(\log n) \end{aligned}$$

## 7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- ▶ **Insert( $x$ )**: insert element  $x$ .
- ▶ **Search( $k$ )**: search for element with key  $k$ .
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Augment an existing data-structure instead of developing a new one.

## 7.4 Augmenting Data Structures

Suppose you want to develop a data structure with:

- ▶ **Insert( $x$ )**: insert element  $x$ .
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### How to augment a data-structure

1. choose an underlying data-structure
2. determine additional information to be stored in the underlying structure
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**Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time  $\mathcal{O}(\log n)$ .**

1. We choose a red-black tree as the underlying data-structure.
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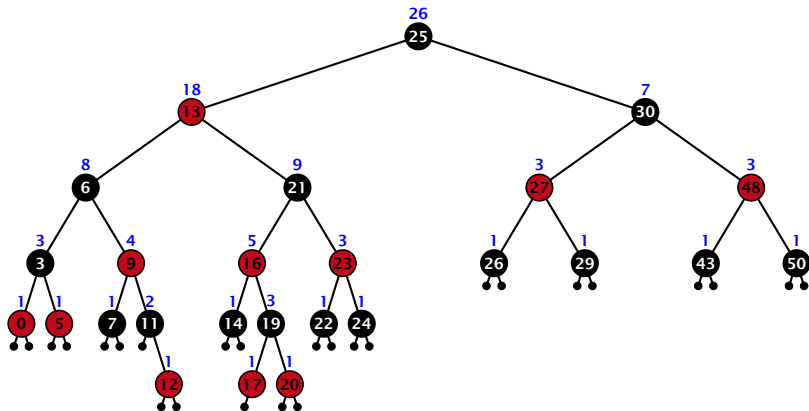
4. How does find-by-rank work?

Find-by-rank( $k$ ) := Select( $\text{root}, k$ ) with

### Algorithm 7 Select( $x, i$ )

```
1: if  $x = \text{null}$  then return error
2: if  $\text{left}[x] \neq \text{null}$  then  $r \leftarrow \text{left}[x].\text{size} + 1$  else  $r \leftarrow 1$ 
3: if  $i = r$  then return  $x$ 
4: if  $i < r$  then
5:     return Select( $\text{left}[x], i$ )
6: else
7:     return Select( $\text{right}[x], i - r$ )
```

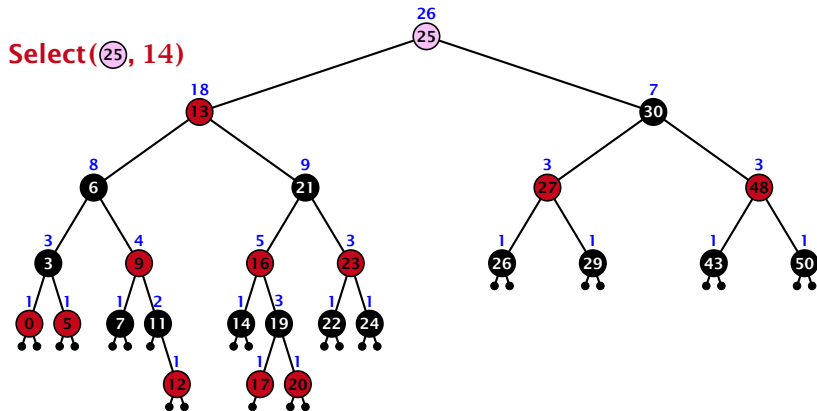
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### Find-by-rank:

- ▶ decide whether you have to proceed into the left or right sub-tree
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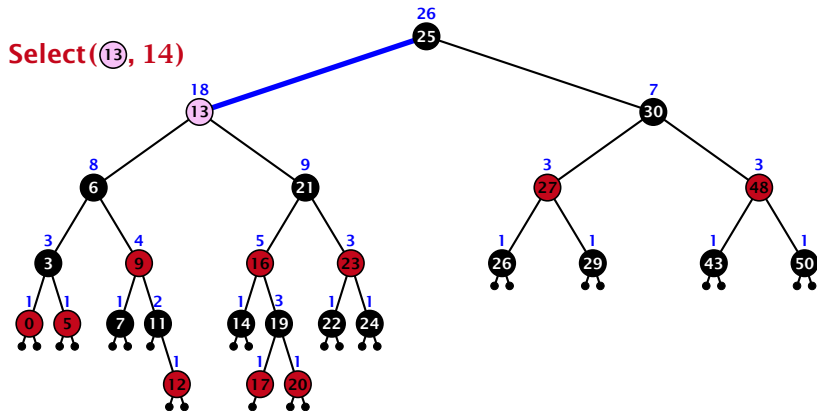
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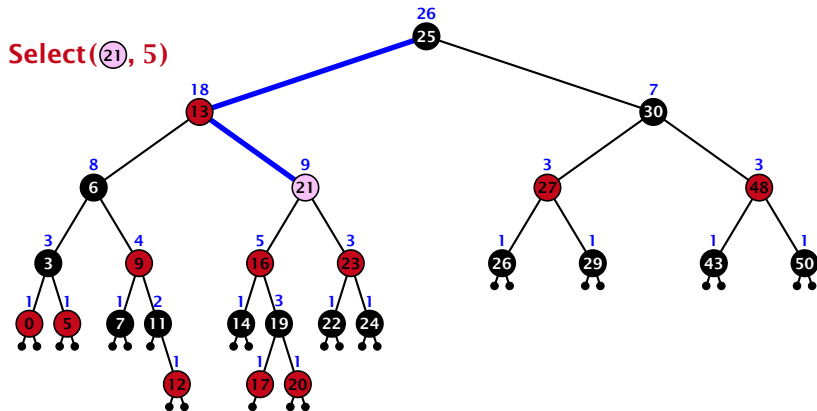
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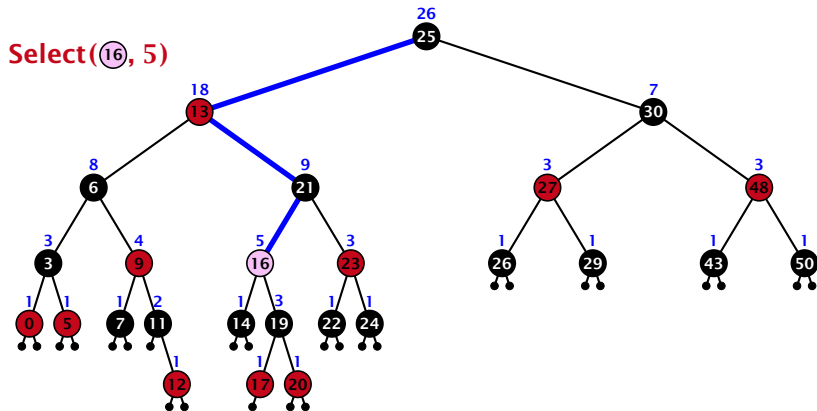
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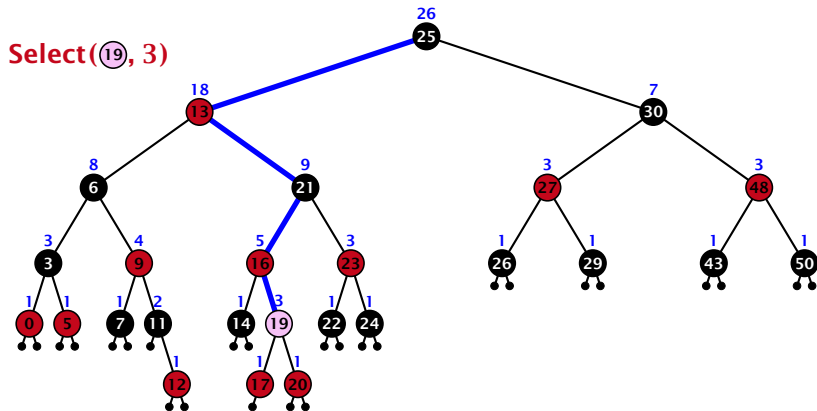


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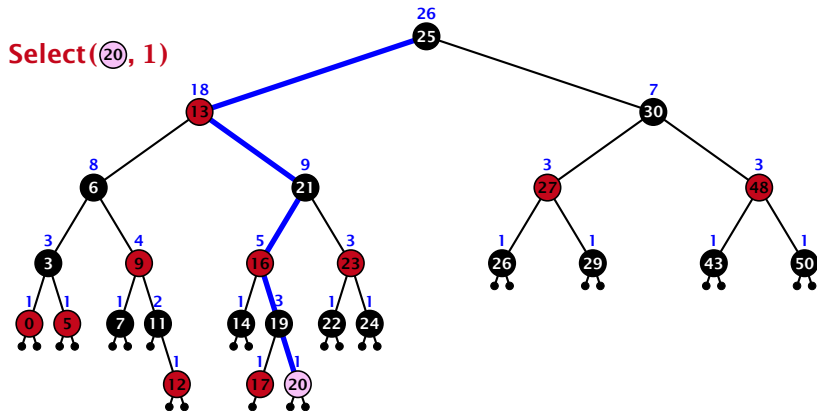
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**Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time  $\mathcal{O}(\log n)$ .**

3. How do we maintain information?

**Search( $k$ ):** Nothing to do.

**Insert( $x$ ):** When going down the search path increase the size field for each visited node. Maintain the size field during rotations.

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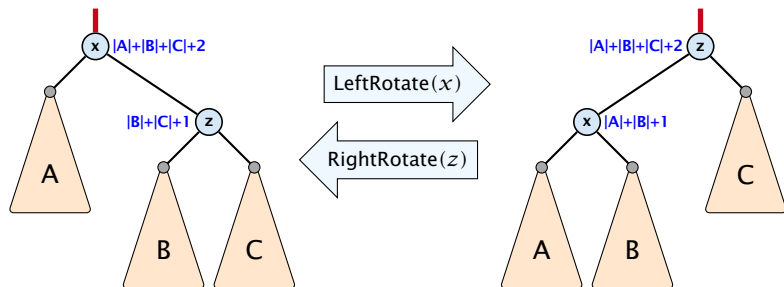
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# Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:



The nodes  $x$  and  $z$  are the only nodes changing their size-fields.

The new size-fields can be computed **locally** from the size-fields of the children.

## 7.5 $(a, b)$ -trees

### Definition 7

For  $b \geq 2a - 1$  an  $(a, b)$ -tree is a search tree with the following properties

1. all leaves have the same distance to the root
2. every internal non-root vertex  $v$  has at least  $a$  and at most  $b$  children
3. the root has degree at least 2 if the tree is non-empty
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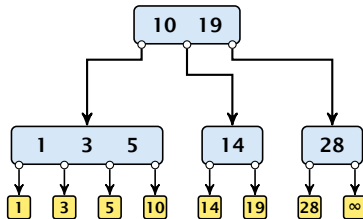
Each internal node  $v$  with  $d(v)$  children stores  $d - 1$  keys  $k_1, \dots, k_{d-1}$ . The  $i$ -th subtree of  $v$  fulfills

$$k_{i-1} < \text{key in } i\text{-th sub-tree} \leq k_i ,$$

where we use  $k_0 = -\infty$  and  $k_d = \infty$ .

## 7.5 (a, b)-trees

### Example 8



## 7.5 ( $a, b$ )-trees

### Variants

- ▶ The dummy leaf element may not exist; it only makes implementation more convenient.
- ▶ Variants in which  $b = 2a$  are commonly referred to as  $B$ -trees.
- ▶ A  $B$ -tree usually refers to the variant in which keys and data are stored at internal nodes.
- ▶ A  $B^+$  tree stores the data only at leaf nodes as in our definition. Sometimes the leaf nodes are also connected in a linear list data structure to speed up the computation of successors and predecessors.
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Let  $T$  be an  $(a, b)$ -tree for  $n > 0$  elements (i.e.,  $n + 1$  leaf nodes) and height  $h$  (number of edges from root to a leaf vertex). Then

1.  $2a^{h-1} \leq n + 1 \leq b^h$
2.  $\log_b(n + 1) \leq h \leq 1 + \log_a\left(\frac{n+1}{2}\right)$

Proof.

Since the root has degree at least  $a$  and all other nodes have degree at least  $a$ , this gives that there are at least  $2a^{h-1}$  leaf nodes. Since the root has degree at most  $b$  and all other nodes have degree at most  $b$ , this gives that there are at most  $b^h$  leaf nodes.



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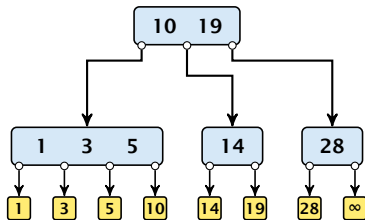
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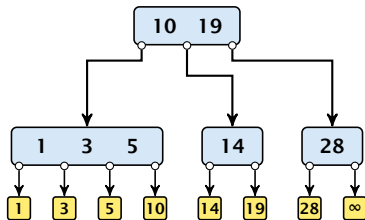


# Search



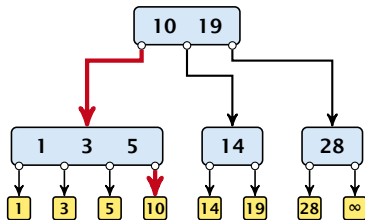
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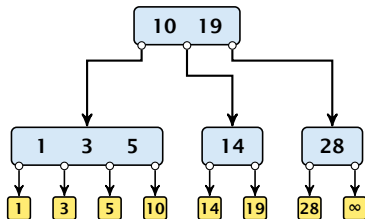
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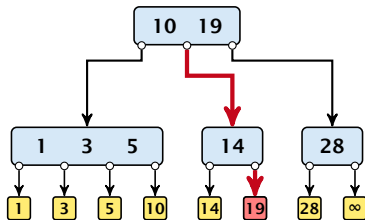
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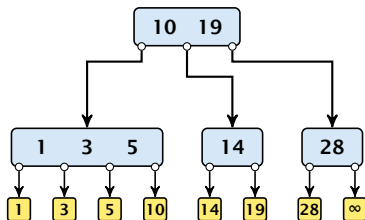


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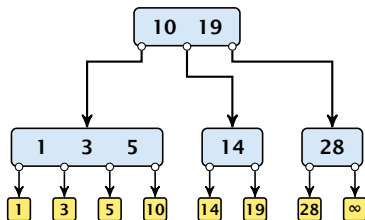


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Time:  $\mathcal{O}(b \cdot h) = \mathcal{O}(b \cdot \log n)$ , if the individual nodes are organized as linear lists.



# Insert

Insert element  $x$ :

- ▶ Follow the path as if searching for  $\text{key}[x]$ .
- ▶ If this search ends in leaf  $\ell$ , insert  $x$  before this leaf.
- ▶ For this add  $\text{key}[x]$  to the key-list of the last internal node  $v$  on the path.
- ▶ If after the insert  $v$  contains  $b$  nodes, do  $\text{Rebalance}(v)$ .

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Rebalance( $v$ ):

- ▶ Let  $k_i$ ,  $i = 1, \dots, b$  denote the keys stored in  $v$ .
- ▶ Let  $j := \lfloor \frac{b+1}{2} \rfloor$  be the middle element.
- ▶ Create two nodes  $v_1$ , and  $v_2$ .  $v_1$  gets all keys  $k_1, \dots, k_{j-1}$  and  $v_2$  gets keys  $k_{j+1}, \dots, k_b$ .
- ▶ Both nodes get at least  $\lfloor \frac{b-1}{2} \rfloor$  keys, and have therefore degree at least  $\lfloor \frac{b-1}{2} \rfloor + 1 \geq a$  since  $b \geq 2a - 1$ .
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- ▶ The key  $k_j$  is promoted to the parent of  $v$ . The current pointer to  $v$  is altered to point to  $v_1$ , and a new pointer (to the right of  $k_j$ ) in the parent is added to point to  $v_2$ .
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- ▶ Let  $k_i$ ,  $i = 1, \dots, b$  denote the keys stored in  $v$ .
- ▶ Let  $j := \lfloor \frac{b+1}{2} \rfloor$  be the middle element.
- ▶ Create two nodes  $v_1$ , and  $v_2$ .  $v_1$  gets all keys  $k_1, \dots, k_{j-1}$  and  $v_2$  gets keys  $k_{j+1}, \dots, k_b$ .
- ▶ Both nodes get at least  $\lfloor \frac{b-1}{2} \rfloor$  keys, and have therefore degree at least  $\lfloor \frac{b-1}{2} \rfloor + 1 \geq a$  since  $b \geq 2a - 1$ .
- ▶ They get at most  $\lceil \frac{b-1}{2} \rceil$  keys, and have therefore degree at most  $\lceil \frac{b-1}{2} \rceil + 1 \leq b$  (since  $b \geq 2$ ).
- ▶ The key  $k_j$  is promoted to the parent of  $v$ . The current pointer to  $v$  is altered to point to  $v_1$ , and a new pointer (to the right of  $k_j$ ) in the parent is added to point to  $v_2$ .
- ▶ Then, re-balance the parent.

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Rebalance( $v$ ):

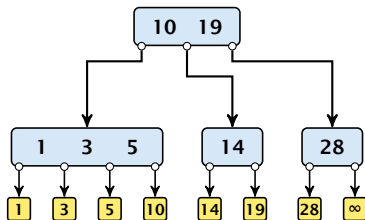
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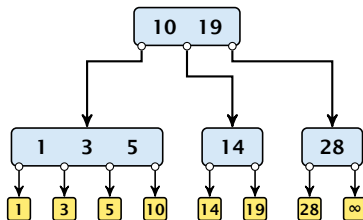
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# Insert



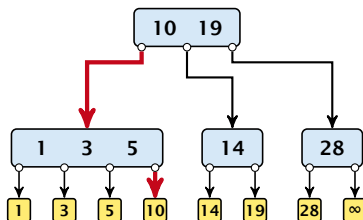
# Insert

## Insert(8)



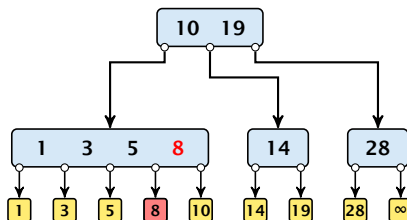
# Insert

## Insert(8)



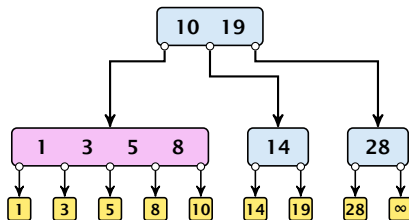
# Insert

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# Insert

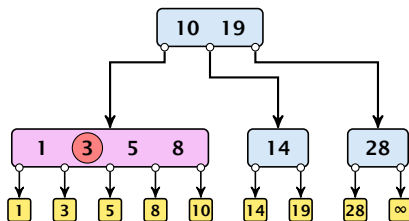
## Insert(8)



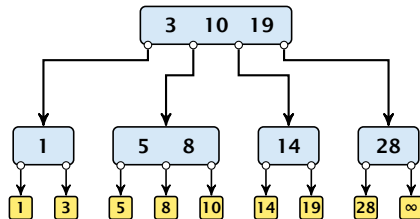


# Insert

Insert(8)

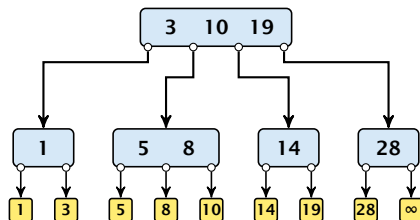


# Insert



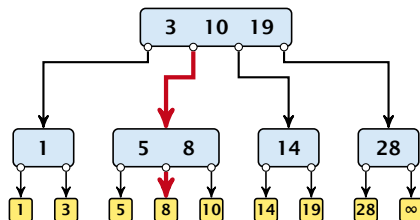
# Insert

## Insert(6)



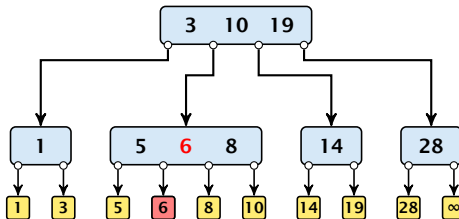
# Insert

## Insert(6)



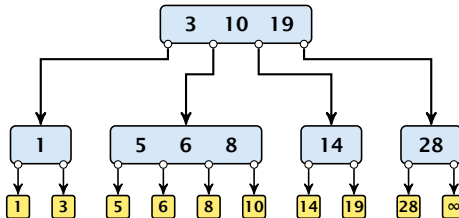
# Insert

## Insert(6)



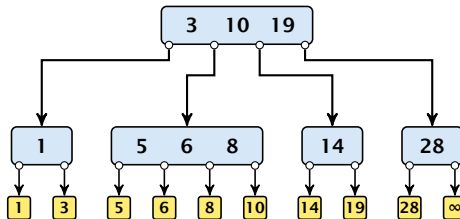
# Insert

## Insert(6)



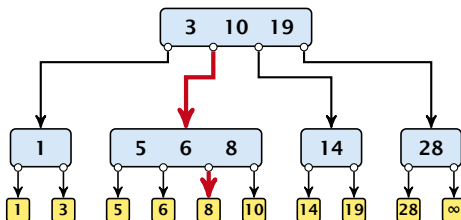
# Insert

## Insert(7)



# Insert

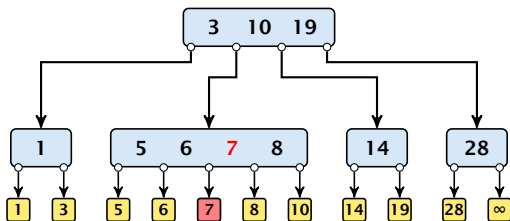
## Insert(7)





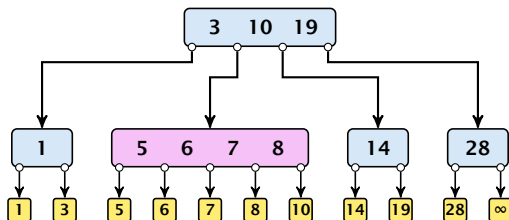
# Insert

## Insert(7)



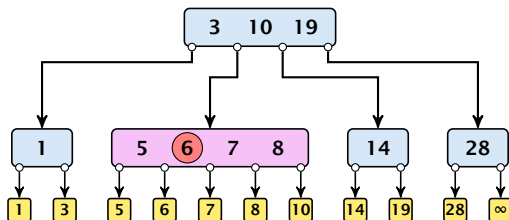
# Insert

Insert(7)



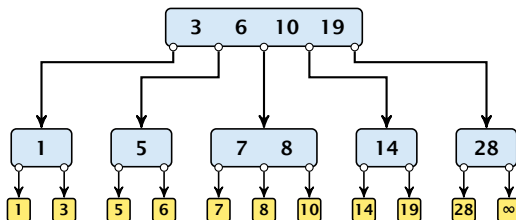
# Insert

Insert(7)



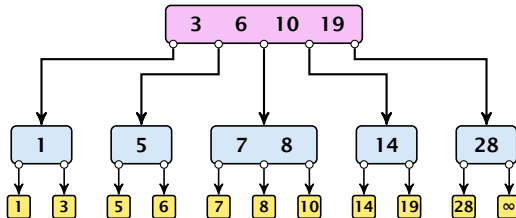
# Insert

## Insert(7)



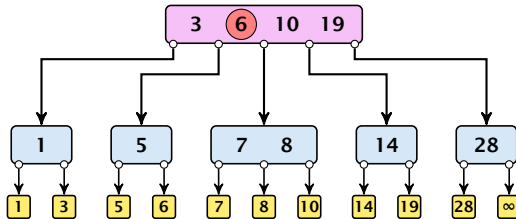
# Insert

## Insert(7)



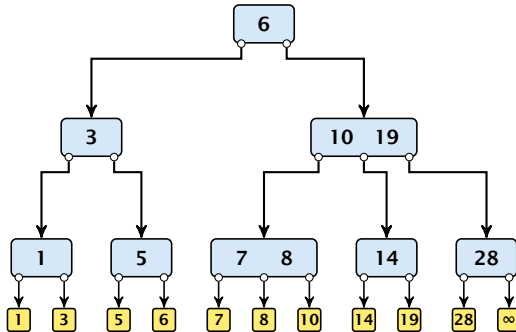
# Insert

Insert(7)



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Insert(7)



# Delete

Delete element  $x$  (pointer to leaf vertex):

- ▶ Let  $v$  denote the parent of  $x$ . If  $\text{key}[x]$  is contained in  $v$ , remove the key from  $v$ , and delete the leaf vertex.
- ▶ Otherwise delete the key of the predecessor of  $x$  from  $v$ ; delete the leaf vertex; and replace the occurrence of  $\text{key}[x]$  in internal nodes by the predecessor key. (Note that it appears in exactly one internal vertex).
- ▶ If now the number of keys in  $v$  is below  $a - 1$  perform  $\text{Rebalance}'(v)$ .



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- ▶ If now the number of keys in  $v$  is below  $a - 1$  perform  $\text{Rebalance}'(v)$ .

# Delete

Rebalance' ( $v$ ):

- ▶ If there is a neighbour of  $v$  that has at least  $a$  keys take over the largest (if right neighbor) or smallest (if left neighbour) and the corresponding sub-tree.
- ▶ If not: merge  $v$  with one of its neighbours.
- ▶ The merged node contains at most  $(a - 2) + (a - 1) + 1$  keys, and has therefore at most  $2a - 1 \leq b$  successors.
- ▶ Then rebalance the parent.
- ▶ During this process the root may become empty. In this case the root is deleted and the height of the tree decreases.

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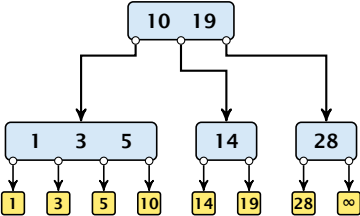
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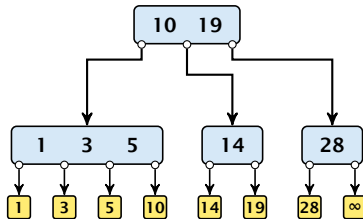
# Delete





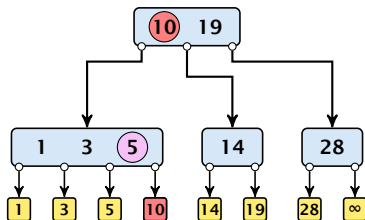
# Delete

Delete(10)



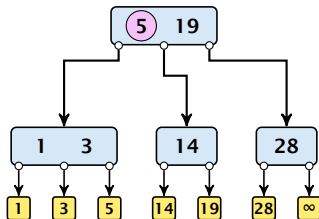
# Delete

Delete(10)

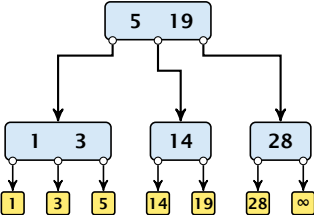


# Delete

Delete(10)

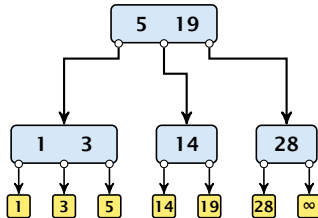


# Delete



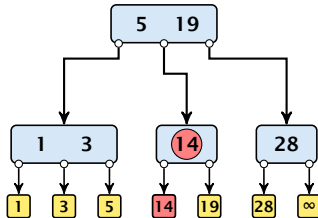
# Delete

## Delete(14)



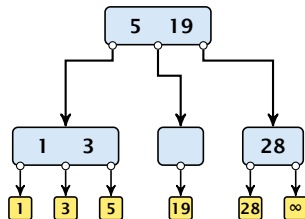
# Delete

Delete(14)



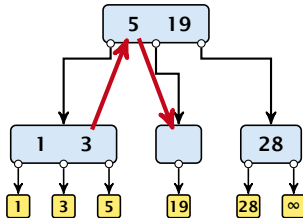
# Delete

Delete(14)



# Delete

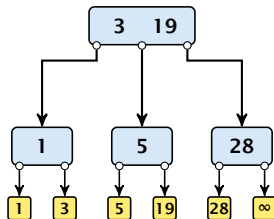
Delete(14)



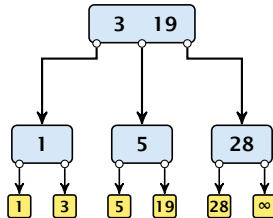


# Delete

Delete(14)

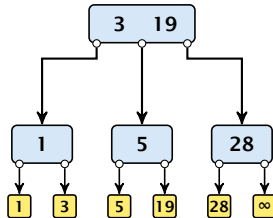


# Delete



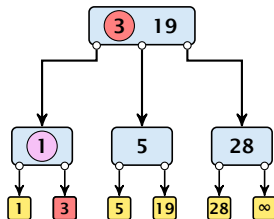
# Delete

## Delete(3)



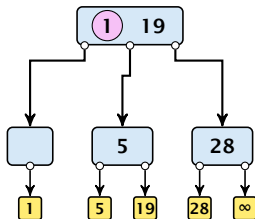
# Delete

## Delete(3)



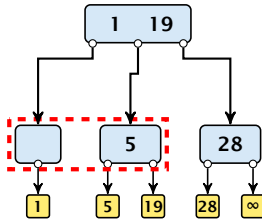
# Delete

## Delete(3)



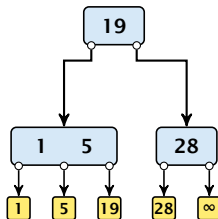
# Delete

## Delete(3)

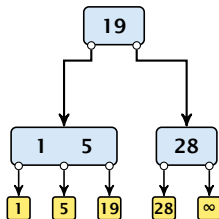


# Delete

## Delete(3)



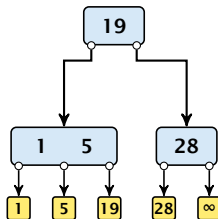
# Delete





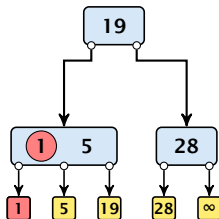
# Delete

## Delete(1)



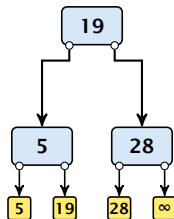
# Delete

## Delete(1)

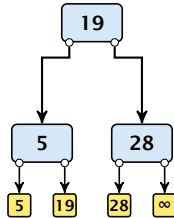


# Delete

## Delete(1)

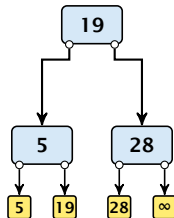


# Delete



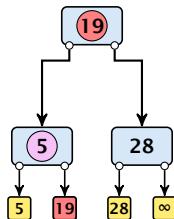
# Delete

Delete(19)



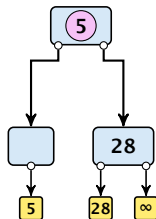
# Delete

Delete(19)



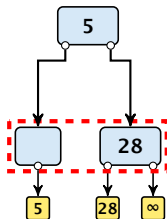
# Delete

Delete(19)



# Delete

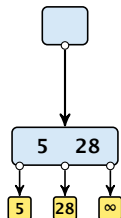
## Delete(19)





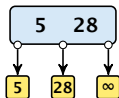
# Delete

## Delete(19)



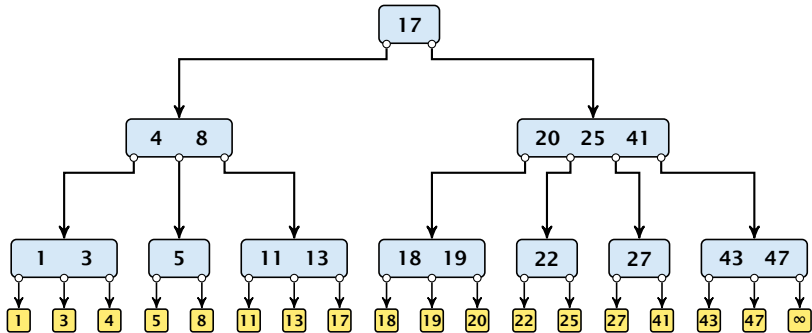
# Delete

## Delete(19)



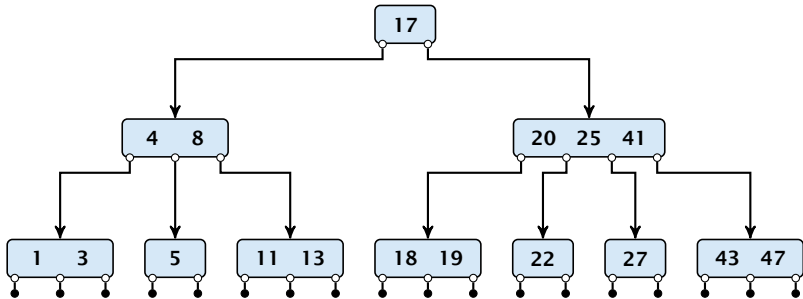
# (2, 4)-trees and red black trees

There is a close relation between red-black trees and (2,4)-trees:



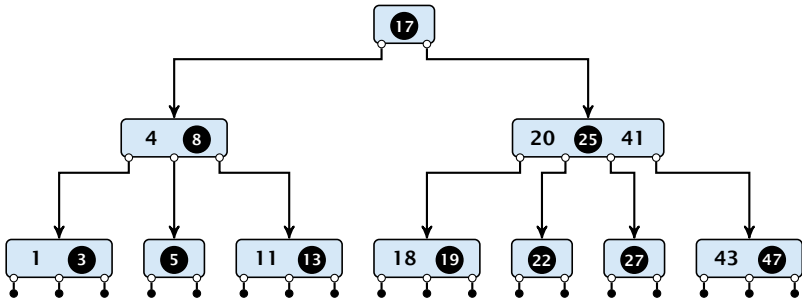
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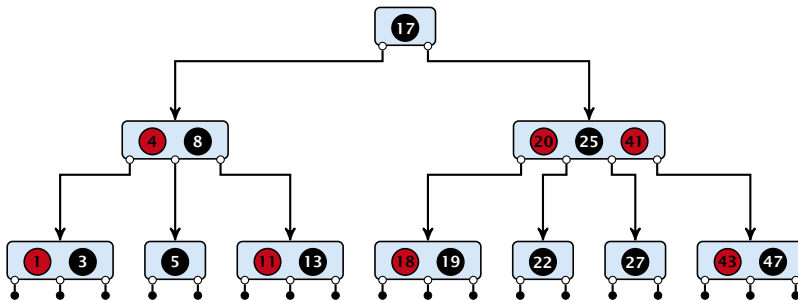
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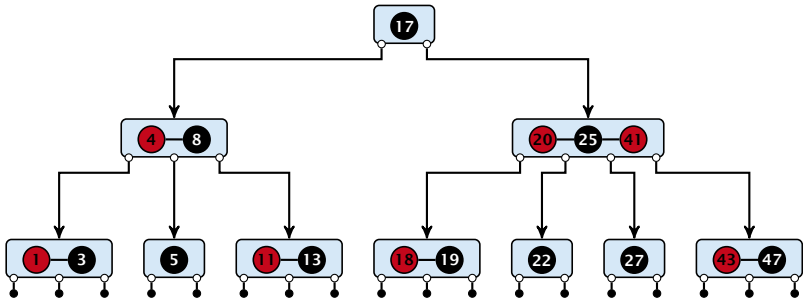
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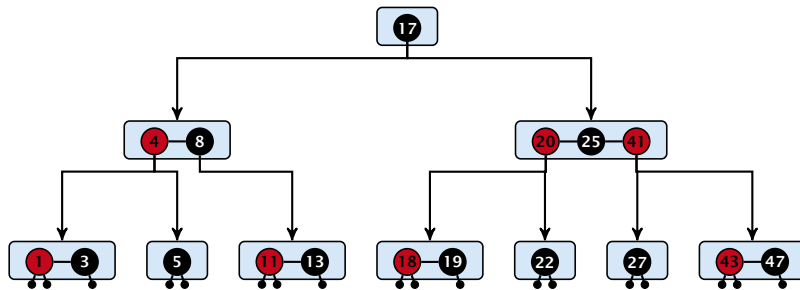
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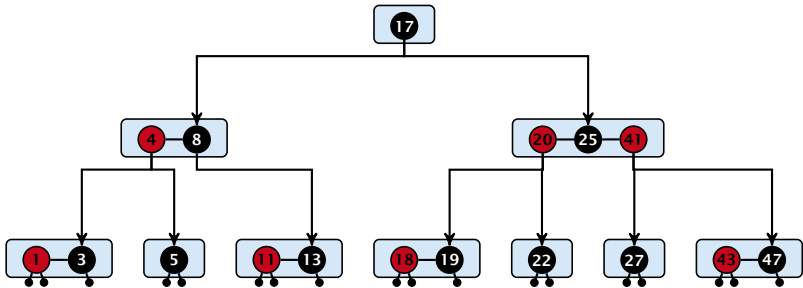
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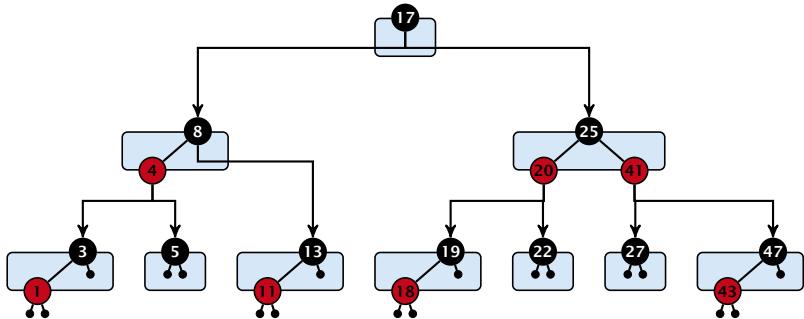
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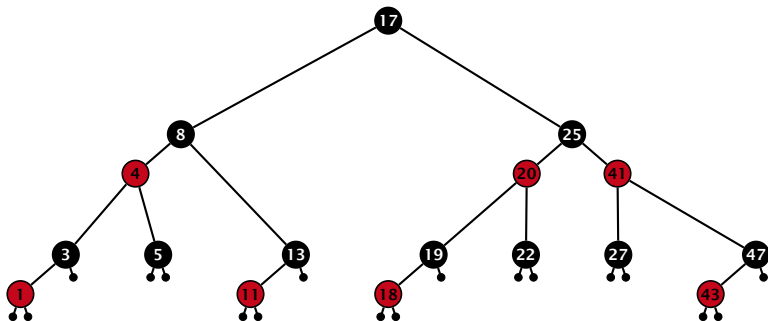
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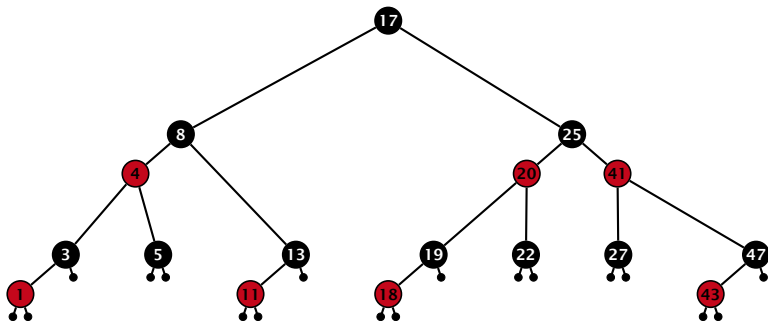
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Note that this correspondence is not unique. In particular, there are different red-black trees that correspond to the same (2, 4)-tree.

## 7.6 Skip Lists

### Why do we not use a list for implementing the ADT Dynamic Set?

- ▶ time for search  $\Theta(n)$
- ▶ time for insert  $\Theta(n)$  (dominated by searching the item)
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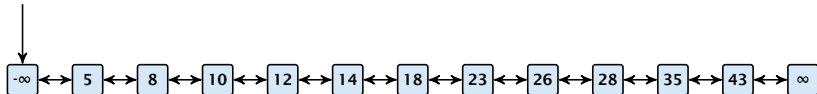
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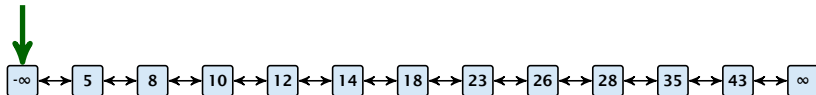
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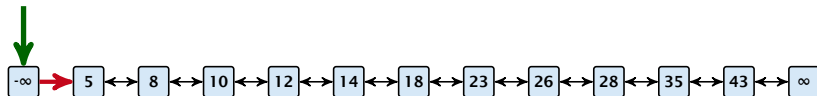




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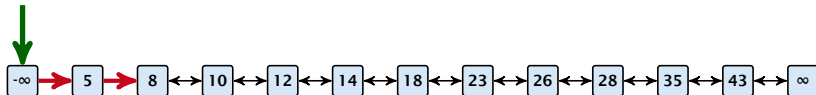
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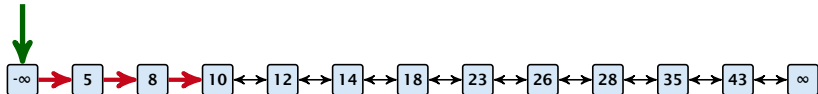
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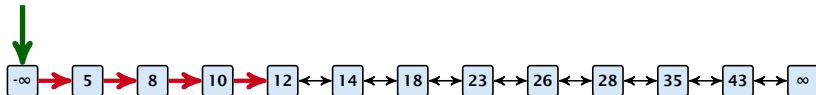
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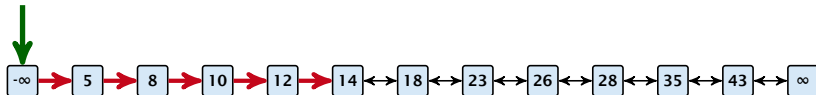
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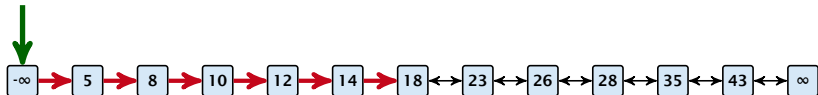
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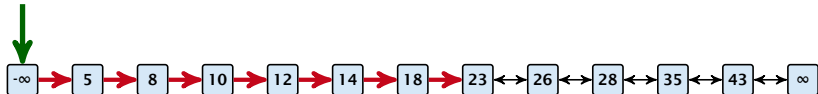
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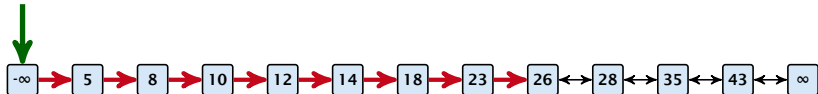
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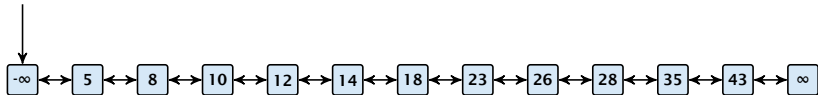
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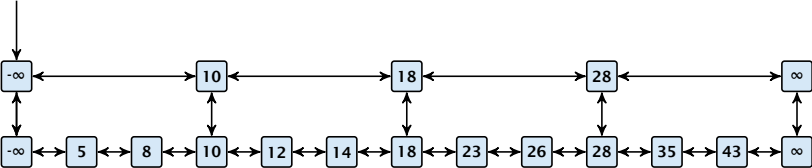
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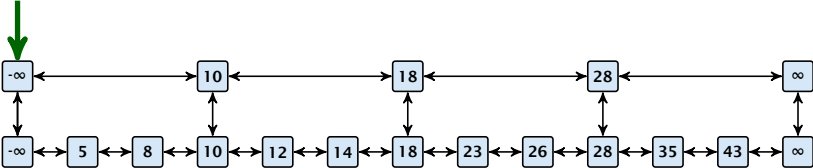
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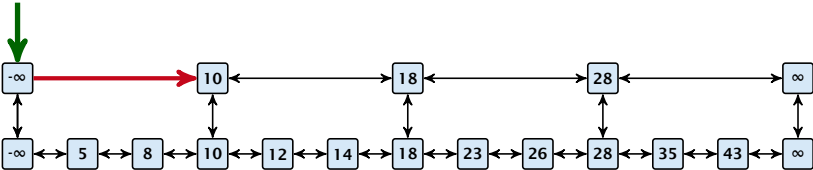
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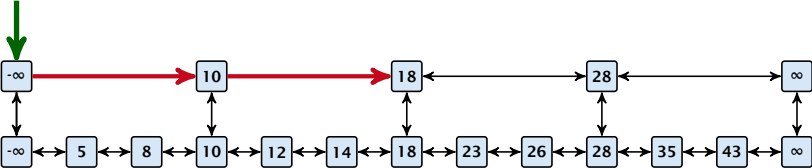
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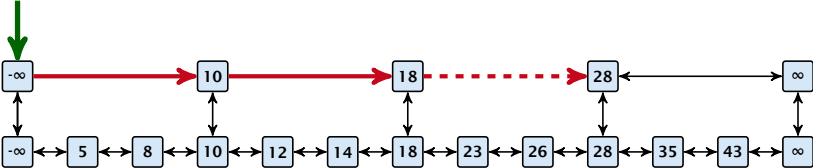
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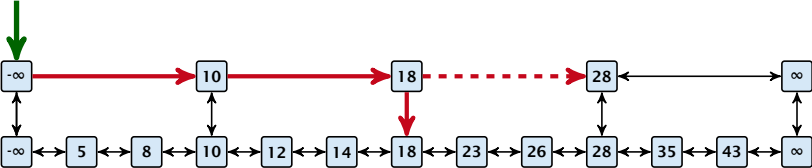




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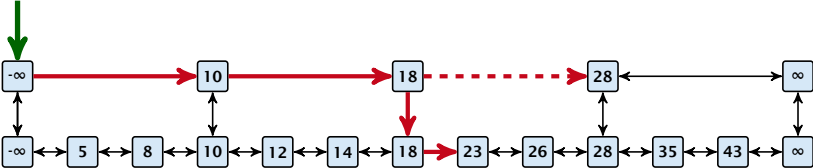
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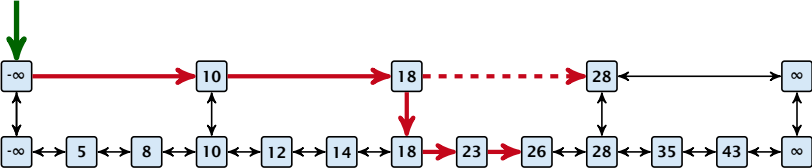
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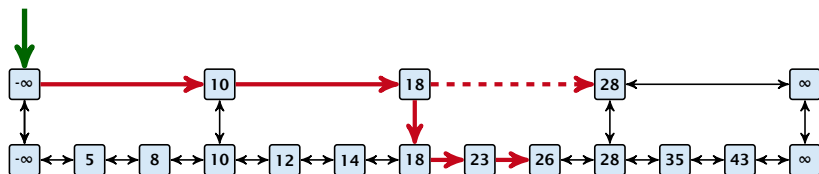
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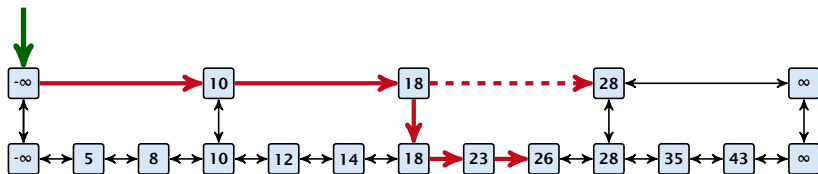


Let  $|L_1|$  denote the number of elements in the “express lane”, and  $|L_0| = n$  the number of all elements (ignoring dummy elements).

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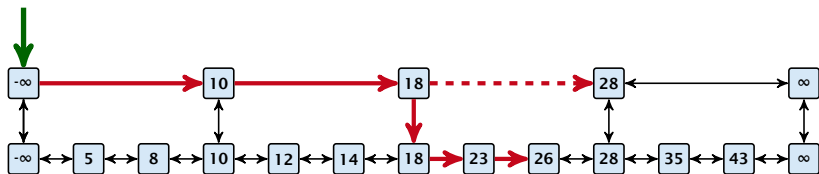
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Choose  $|L_1| = \sqrt{n}$ . Then search time  $\Theta(\sqrt{n})$ .

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Choosing  $k = \Theta(\log n)$  gives a logarithmic running time.

## 7.6 Skip Lists

### How to do insert and delete?

The cost of insert and delete is proportional to the number of elements in the list. Insert or delete may require a lot of reorganization.

Use randomization instead!

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### How to do insert and delete?

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### Insert:

- ▶ A search operation gives you the insert position for element  $x$  in every list.
- ▶ Flip a coin until it shows head, and record the number  $t \in \{1, 2, \dots\}$  of trials needed.
- ▶ Insert  $x$  into lists  $L_0, \dots, L_{t-1}$ .

### Delete:

You get all predecessors via backward pointers.

Delete  $x$  in all lists it actually appears in.

The time for both operations is dominated by the search time.



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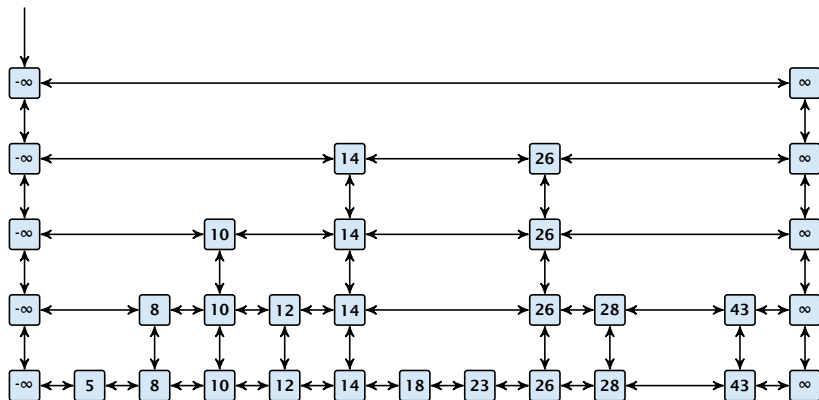
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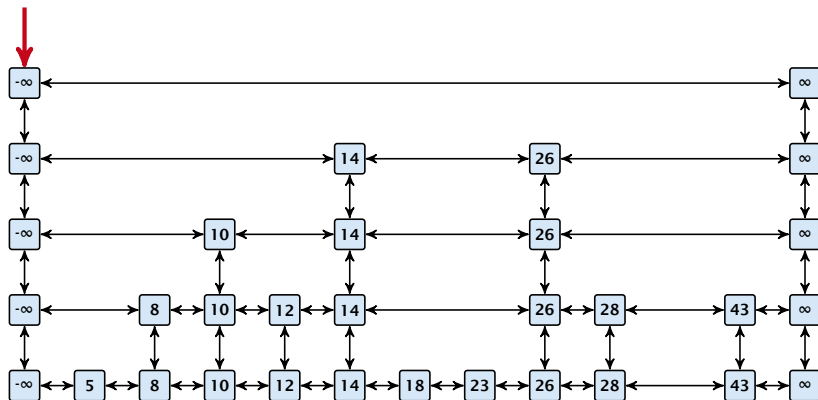
Insert (35):





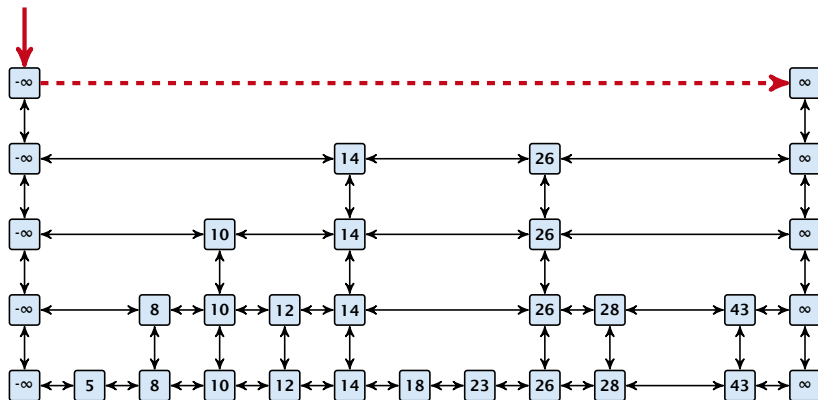
## 7.6 Skip Lists

Insert (35):



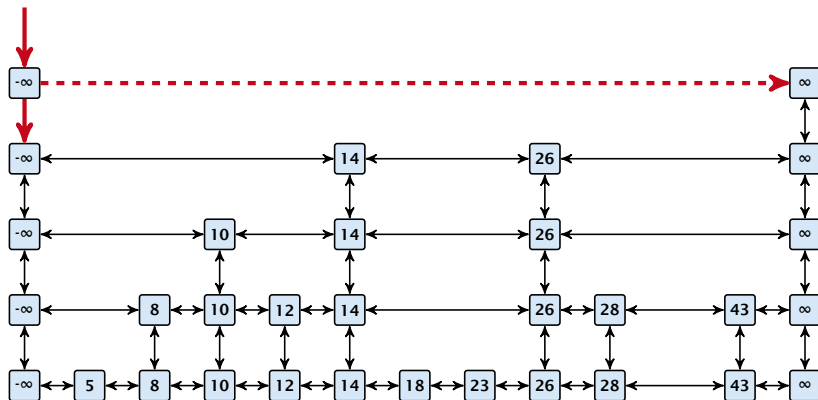
## 7.6 Skip Lists

Insert (35):



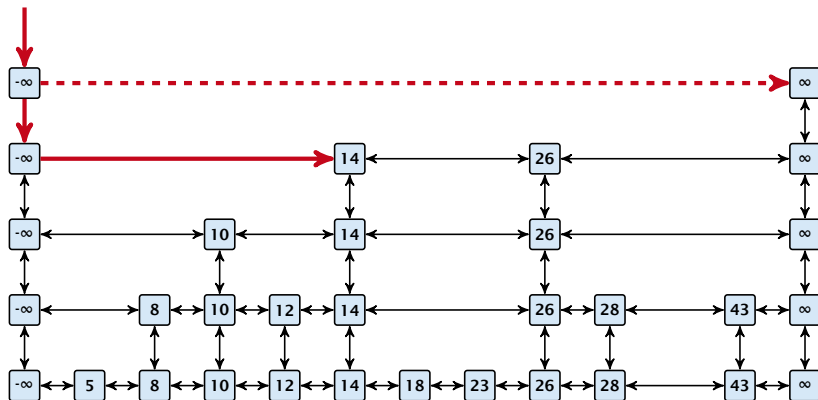
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Insert (35):



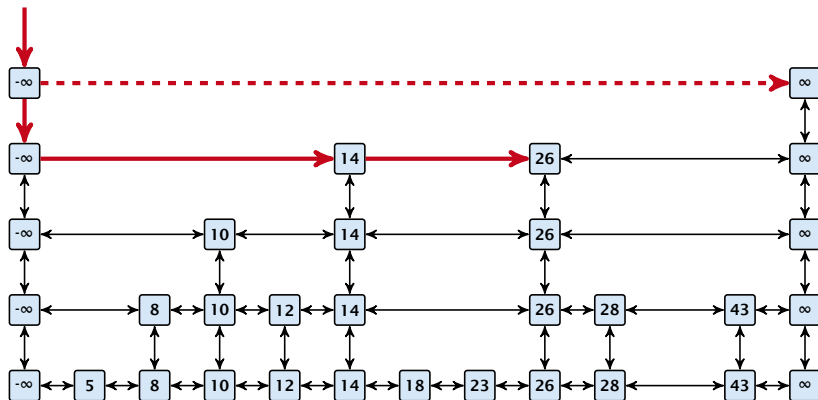
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Insert (35):



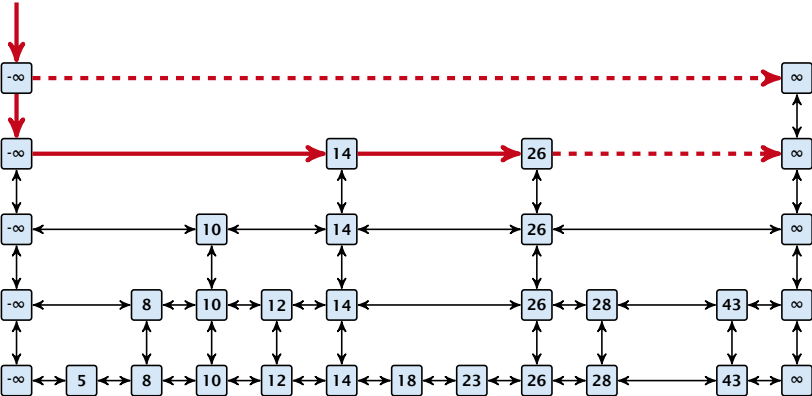
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Insert (35):



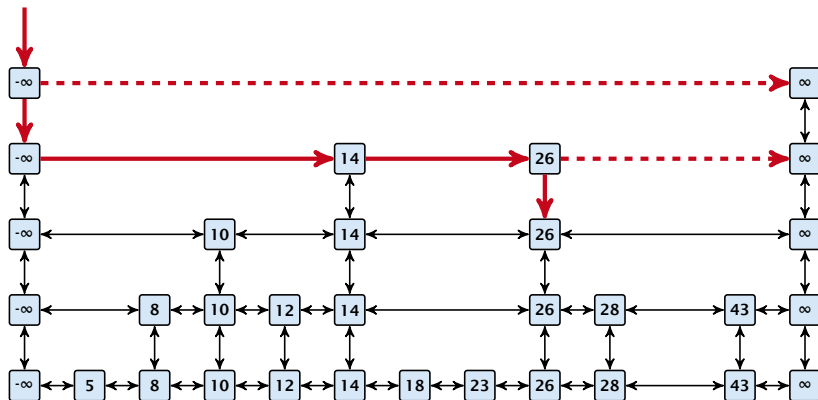
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Insert (35):



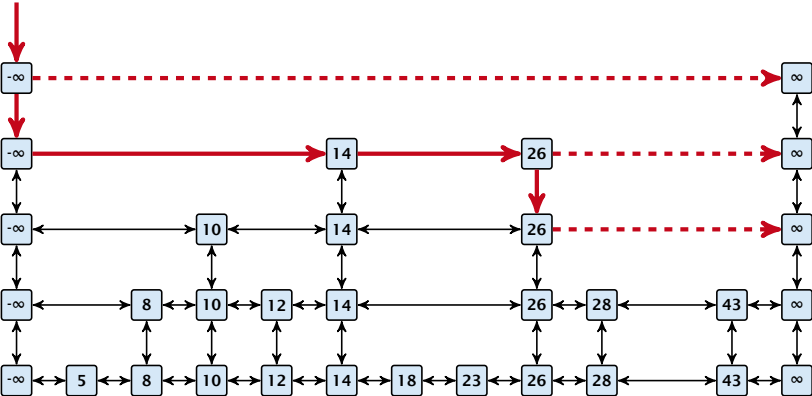
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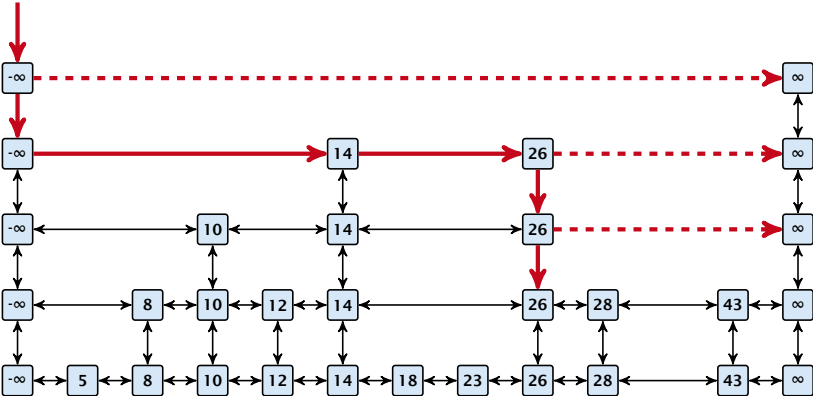
Insert (35):





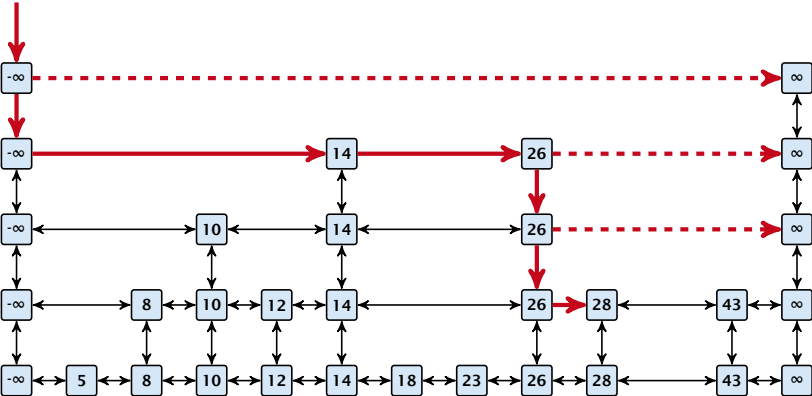
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Insert (35):



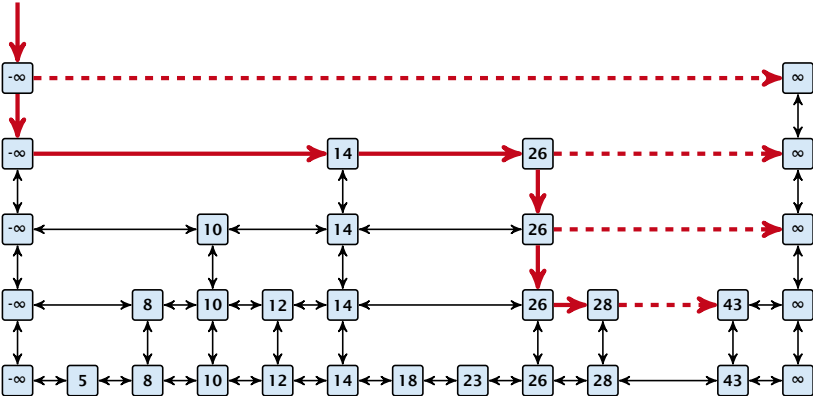
# 7.6 Skip Lists

Insert (35):



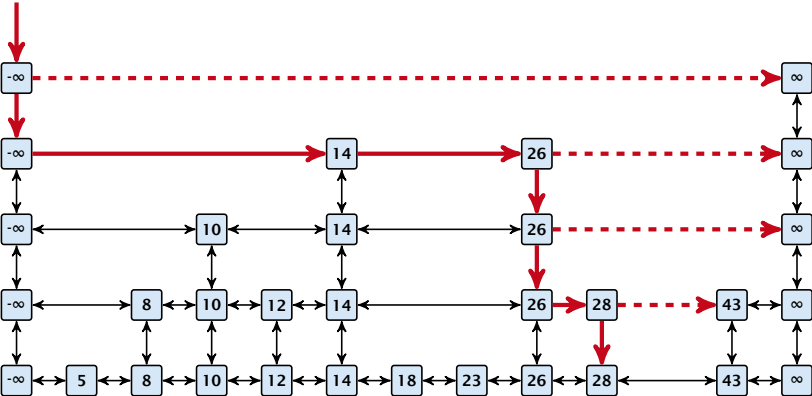
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Insert (35):



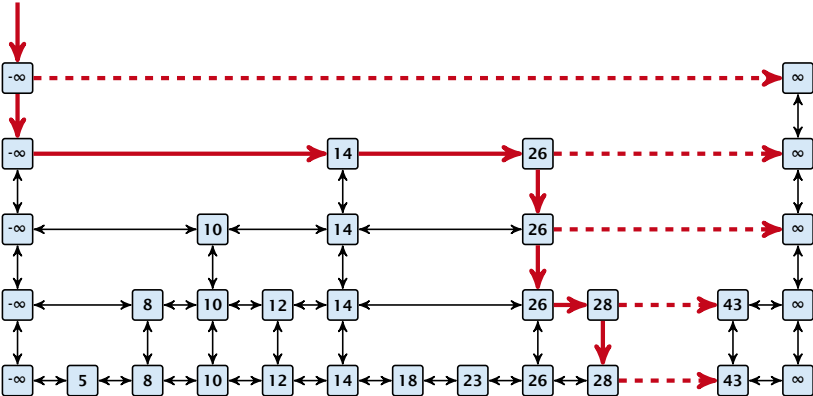
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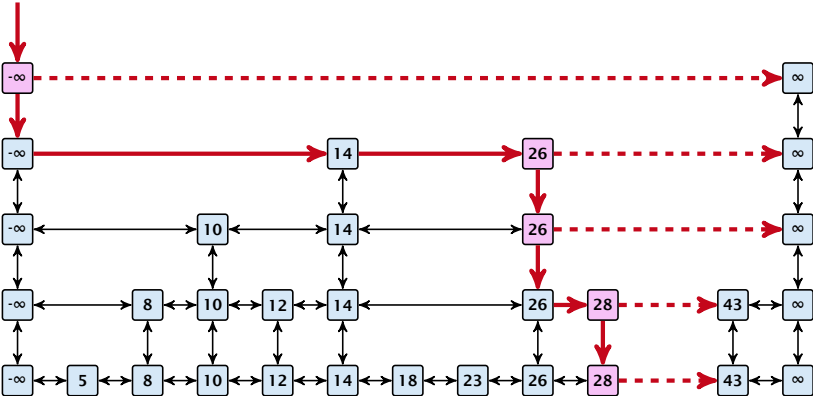
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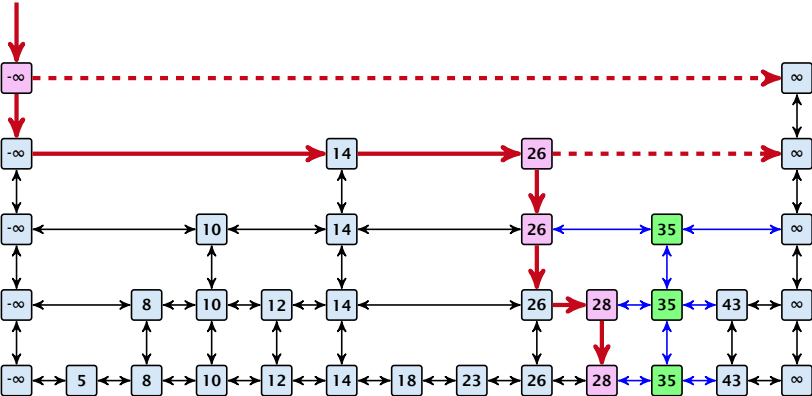
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# High Probability

## Definition 10 (High Probability)

We say a **randomized** algorithm has running time  $\mathcal{O}(\log n)$  with **high probability** if for any constant  $\alpha$  the running time is at most  $\mathcal{O}(\log n)$  with probability at least  $1 - \frac{1}{n^\alpha}$ .

Here the  $\mathcal{O}$ -notation hides a constant that may depend on  $\alpha$ .



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# High Probability

Suppose there are a **polynomially** many events  $E_1, E_2, \dots, E_\ell$ ,  $\ell = n^c$  each holding with high probability (e.g.  $E_i$  may be the event that the  $i$ -th search in a skip list takes time at most  $\mathcal{O}(\log n)$ ).

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This means  $\Pr[E_1 \wedge \dots \wedge E_\ell]$  holds with high probability.

## 7.6 Skip Lists

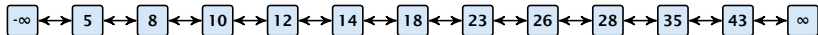
### Lemma 11

*A search (and, hence, also insert and delete) in a skip list with  $n$  elements takes time  $\mathcal{O}(\log n)$  with high probability (w. h. p.).*



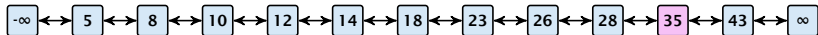
## 7.6 Skip Lists

Backward analysis:



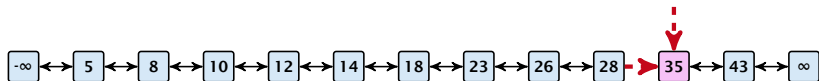
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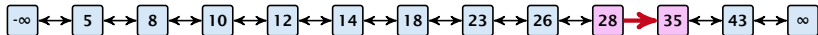
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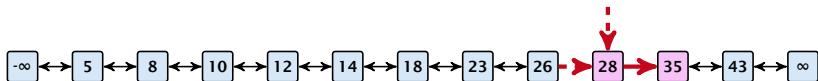
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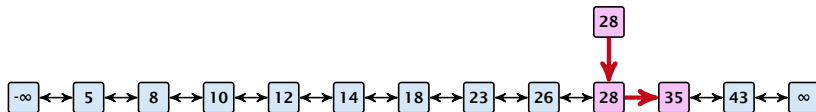
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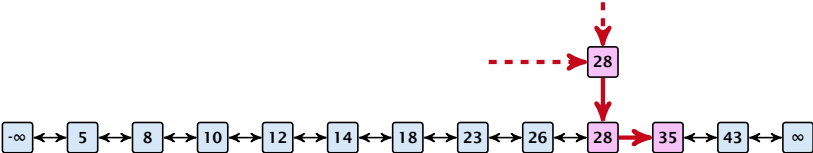
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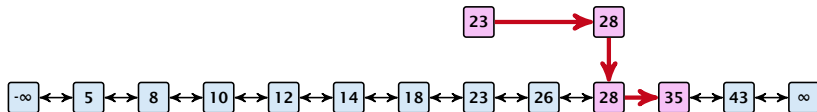
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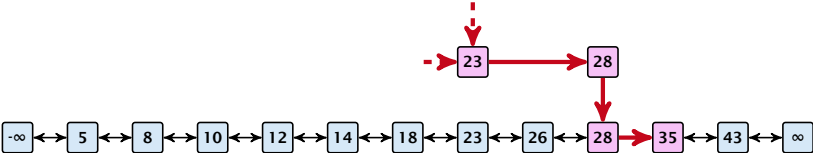
Backward analysis:





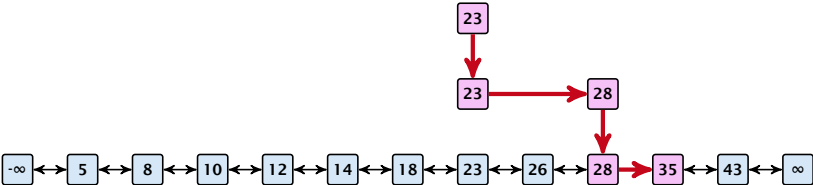
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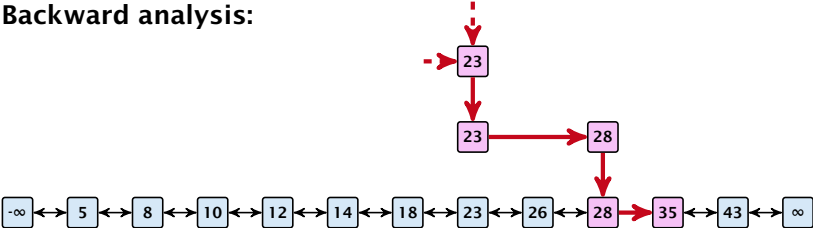
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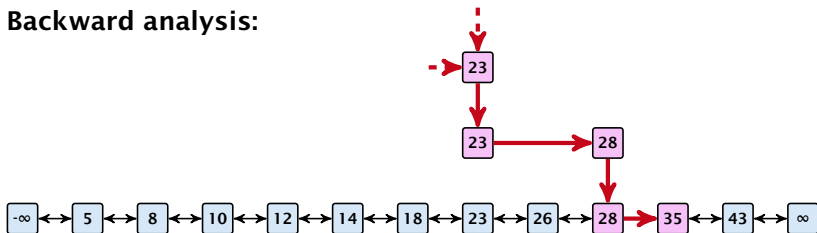
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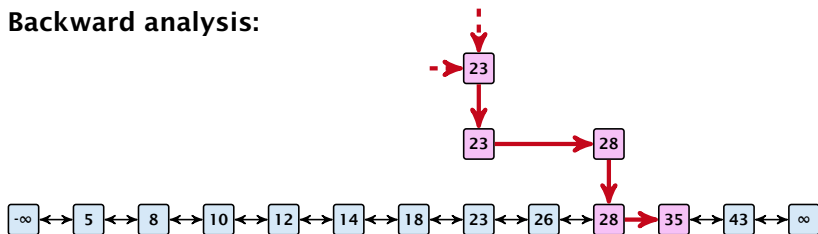
Backward analysis:



At each point the path goes up with probability  $1/2$  and left with probability  $1/2$ .

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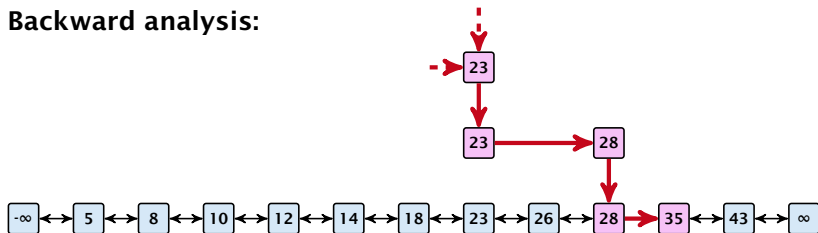
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We show that w.h.p:

- ▶ A “long” search path must also go very high.

## 7.6 Skip Lists

Backward analysis:



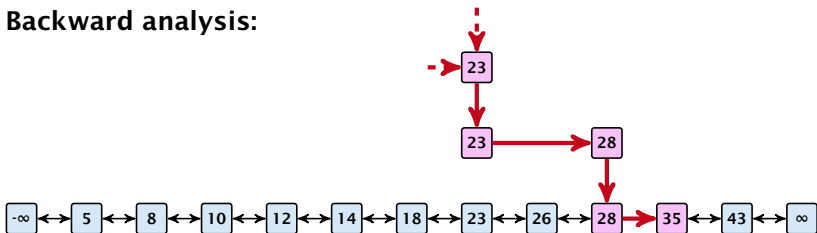
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At each point the path goes up with probability  $1/2$  and left with probability  $1/2$ .

We show that w.h.p.:

- ▶ A “long” search path must also go very high.
- ▶ There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$



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## 7.6 Skip Lists

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In particular, this means that during the construction in the backward analysis we see at most  $k$  heads (i.e., coin flips that tell you to go up) in  $z$  trials.

## 7.6 Skip Lists

$$\Pr[E_{z,k}]$$

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## 7.6 Skip Lists

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## 7.6 Skip Lists

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for  $\alpha \geq 1$ .

## 7.6 Skip Lists

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This means, the search requires at most  $z$  steps, w. h. p.

## 7.7 Hashing

### Dictionary:

- ▶  **$S.insert(x)$** : Insert an element  $x$ .
- ▶  **$S.delete(x)$** : Delete the element pointed to by  $x$ .
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So far we have implemented the search for a key by carefully choosing split-elements.

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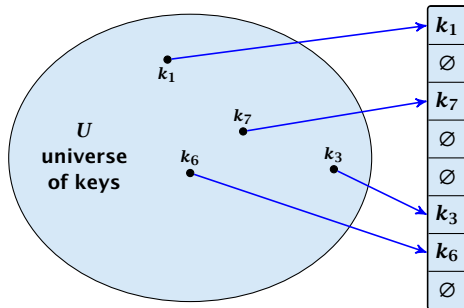
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# Direct Addressing

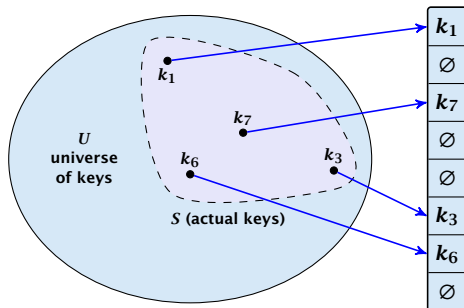
Ideally the hash function maps **all** keys to different memory locations.



This special case is known as **Direct Addressing**. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

# Perfect Hashing

Suppose that we **know** the set  $S$  of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Such a hash function  $h$  is called a **perfect hash function** for set  $S$ .

# Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

## Problem: Collisions

Usually the universe  $U$  is much larger than the table-size  $n$ .

Hence, there may be two elements  $k_1, k_2$  from the set  $S$  that map to the same memory location (i.e.,  $h(k_1) = h(k_2)$ ). This is called a **collision**.

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Typically, collisions do not appear once the size of the set  $S$  of actual keys gets close to  $n$ , but already when  $|S| \geq \omega(\sqrt{n})$ .

## Lemma 12

*The probability of having a collision when hashing  $m$  elements into a table of size  $n$  under uniform hashing is at least*

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}}.$$

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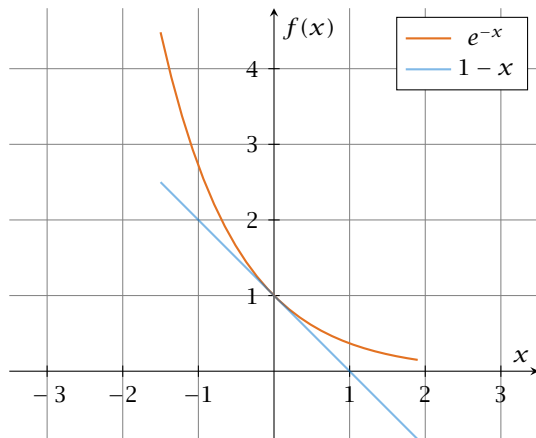
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Here the first equality follows since the  $\ell$ -th element that is hashed has a probability of  $\frac{n-\ell+1}{n}$  to not generate a collision under the condition that the previous elements did not induce collisions. □



# Collisions



The inequality  $1 - x \leq e^{-x}$  is derived by stopping the Taylor-expansion of  $e^{-x}$  after the second term.

# Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- ▶ **open addressing**, aka. closed hashing
- ▶ **hashing with chaining**, aka. closed addressing, open hashing.

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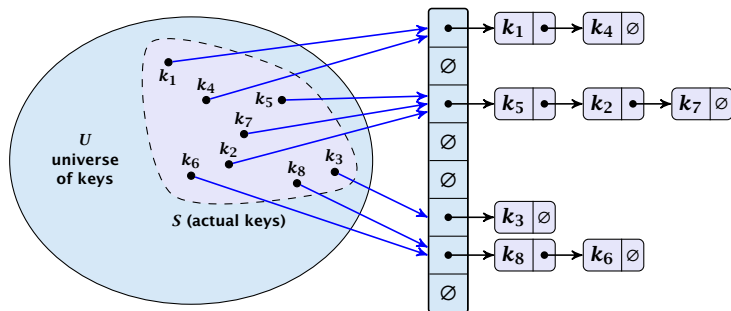
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# Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- ▶ Access: compute  $h(x)$  and search list for  $\text{key}[x]$ .
- ▶ Insert: insert at the front of the list.



# Hashing with Chaining

Let  $A$  denote a strategy for resolving collisions. We use the following notation:

- ▶  $A^+$  denotes the average time for a **successful** search when using  $A$ ;
- ▶  $A^-$  denotes the average time for an **unsuccessful** search when using  $A$ ;
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$$A^- = 1 + \alpha .$$

# Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key  $k$  in the hash-table and ask for the search-time for  $k$ .

This is 1 plus the number of elements that lie before  $k$  in  $k$ 's list.

Let  $k_\ell$  denote the  $\ell$ -th key inserted into the table.

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The expected successful search cost is

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## Hashing with Chaining

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$$\mathbb{E} \left[ \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m X_{ij} \right) \right] = \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m \mathbb{E}[X_{ij}] \right)$$

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$$\begin{aligned} E \left[ \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m X_{ij} \right) \right] &= \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m E[X_{ij}] \right) \\ &= \frac{1}{m} \sum_{i=1}^m \left( 1 + \sum_{j=i+1}^m \frac{1}{n} \right) \\ &= 1 + \frac{1}{mn} \sum_{i=1}^m (m - i) \\ &= 1 + \frac{1}{mn} \left( m^2 - \frac{m(m+1)}{2} \right) \\ &= 1 + \frac{m-1}{2n} \end{aligned}$$

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Hence, the expected cost for a successful search is  $A^+ \leq 1 + \frac{\alpha}{2}$ .

# Hashing with Chaining

## Disadvantages:

- ▶ pointers increase memory requirements
- ▶ pointers may lead to bad cache efficiency

## Advantages:

- ▶ no à priori limit on the number of elements
- ▶ deletion can be implemented efficiently
- ▶ by using balanced trees instead of linked list one can also obtain worst-case guarantees.

# Open Addressing

All objects are stored in the table itself.

Define a function  $h(k, j)$  that determines the table-position to be examined in the  $j$ -th step. The values  $h(k, 0), \dots, h(k, n - 1)$  must form a permutation of  $0, \dots, n - 1$ .

**Search( $k$ ):** Try position  $h(k, 0)$ ; if it is empty your search fails; otw. continue with  $h(k, 1), h(k, 2), \dots$ .

**Insert( $x$ ):** Search until you find an empty slot; insert your element there. If your search reaches  $h(k, n - 1)$ , and this slot is non-empty then your table is full.

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# Open Addressing

Choices for  $h(k, j)$ :

- ▶ **Linear probing:**

$$h(k, i) = h(k) + i \pmod n$$

(sometimes:  $h(k, i) = h(k) + ci \pmod n$ ).

- ▶ Quadratic probing:

$$h(k, i) = h(k) + c_1 i + c_2 i^2 \pmod n.$$

- ▶ Double hashing:

$$h(k, i) = h_1(k) + ih_2(k) \pmod n.$$

For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing  $h_2(k)$  must be relatively prime to  $n$  (teilerfremd); for quadratic probing  $c_1$  and  $c_2$  have to be chosen carefully).

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# Linear Probing

- ▶ Advantage: **Cache-efficiency**. The new probe position is very likely to be in the cache.
- ▶ Disadvantage: **Primary clustering**. Long sequences of occupied table-positions get longer as they have a larger probability to be hit. Furthermore, they can merge forming larger sequences.

## Lemma 13

Let  $L$  be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2} \left( 1 + \frac{1}{1 - \alpha} \right)$$

$$L^- \approx \frac{1}{2} \left( 1 + \frac{1}{(1 - \alpha)^2} \right)$$

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# Quadratic Probing

- ▶ Not as cache-efficient as Linear Probing.
- ▶ **Secondary clustering**: caused by the fact that all keys mapped to the same position have the same probe sequence.

## Lemma 14

*Let  $Q$  be the method of quadratic probing for resolving collisions:*

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

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# Double Hashing

- ▶ Any probe into the hash-table usually creates a cache-miss.

## Lemma 15

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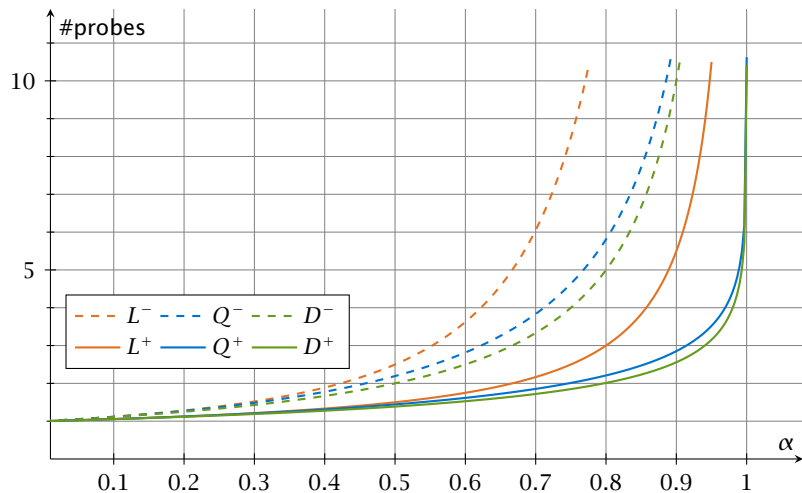
$$D^- \approx \frac{1}{1 - \alpha}$$

# Open Addressing

Some values:

$\alpha$	<i>Linear Probing</i>		<i>Quadratic Probing</i>		<i>Double Hashing</i>	
	$L^+$	$L^-$	$Q^+$	$Q^-$	$D^+$	$D^-$
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

# Open Addressing



# Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- ▶ The probe sequence  $h(k, 0), h(k, 1), h(k, 2), \dots$  is equally likely to be any permutation of  $\langle 0, 1, \dots, n - 1 \rangle$ .

# Analysis of Idealized Open Address Hashing



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$$\Pr[A_1 \cap A_2 \cap \dots \cap A_{i-1}]$$

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$$\Pr[X \geq i]$$

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$$\Pr[X \geq i] = \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \dots \cdot \frac{m-i+2}{n-i+2}$$

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$$\begin{aligned}\Pr[X \geq i] &= \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \dots \cdot \frac{m-i+2}{n-i+2} \\ &\leq \left(\frac{m}{n}\right)^{i-1}\end{aligned}$$

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# Analysis of Idealized Open Address Hashing

$E[X]$



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# Analysis of Idealized Open Address Hashing

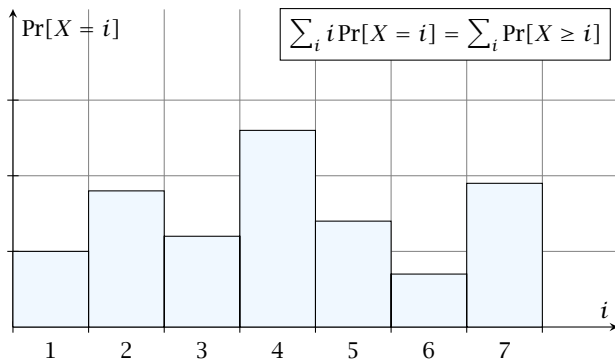
$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \leq \sum_{i=1}^{\infty} \alpha^{i-1} = \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha} .$$

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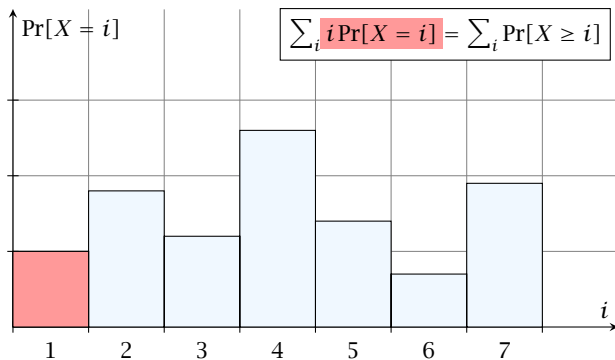
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

# Analysis of Idealized Open Address Hashing



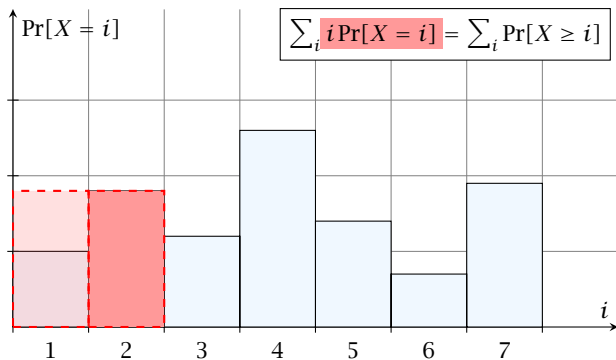
# Analysis of Idealized Open Address Hashing

$i = 1$



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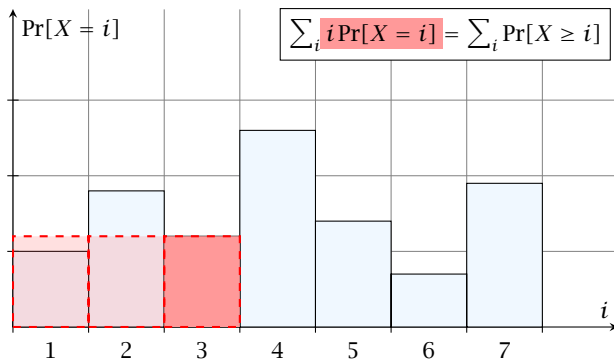
$i = 2$





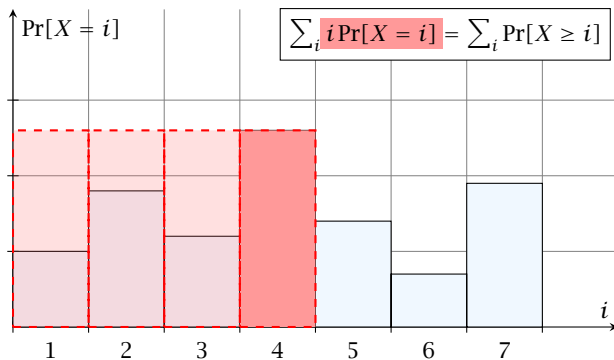
# Analysis of Idealized Open Address Hashing

$i = 3$



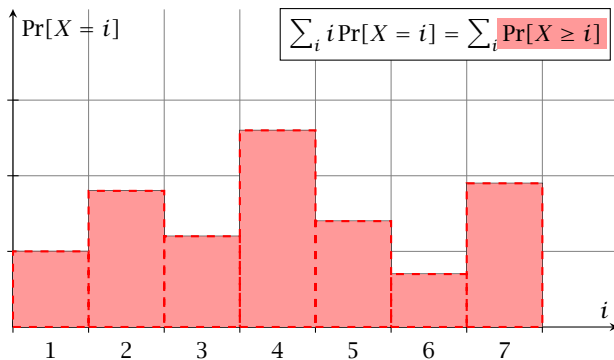
# Analysis of Idealized Open Address Hashing

$i = 4$



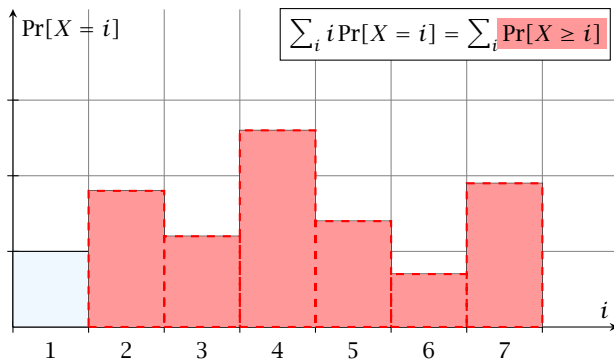
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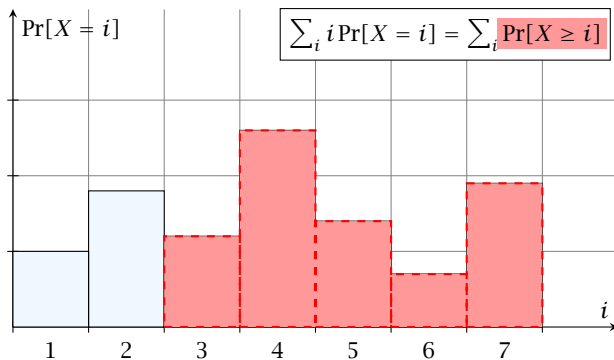
# Analysis of Idealized Open Address Hashing

$i = 2$



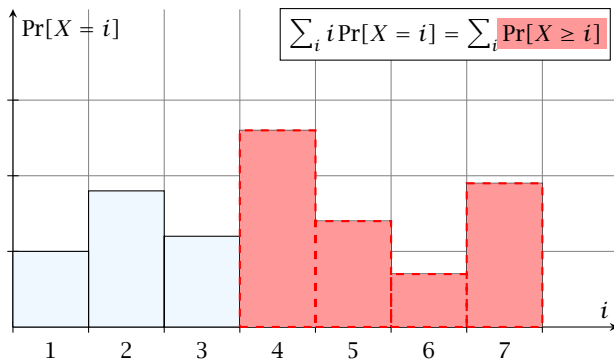
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$i = 3$

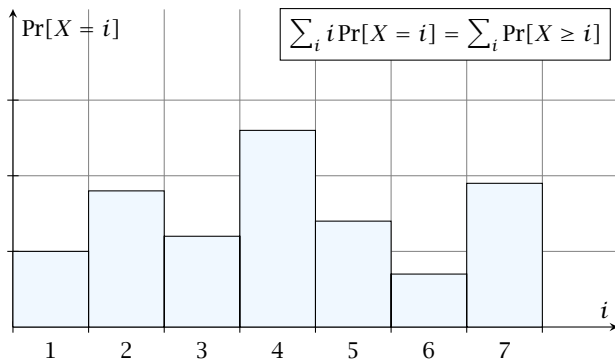


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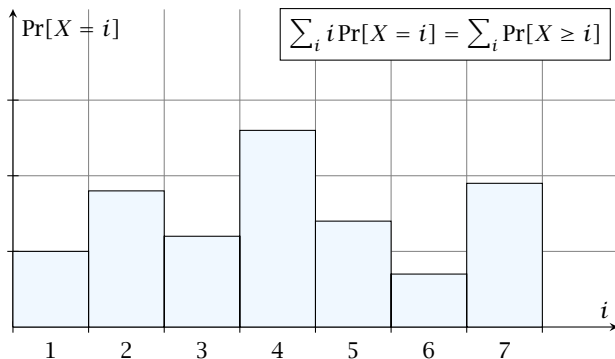
$i = 4$



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The  $j$ -th rectangle appears in both sums  $j$  times. ( $j$  times in the first due to multiplication with  $j$ ; and  $j$  times in the second for summands  $i = 1, 2, \dots, j$ )



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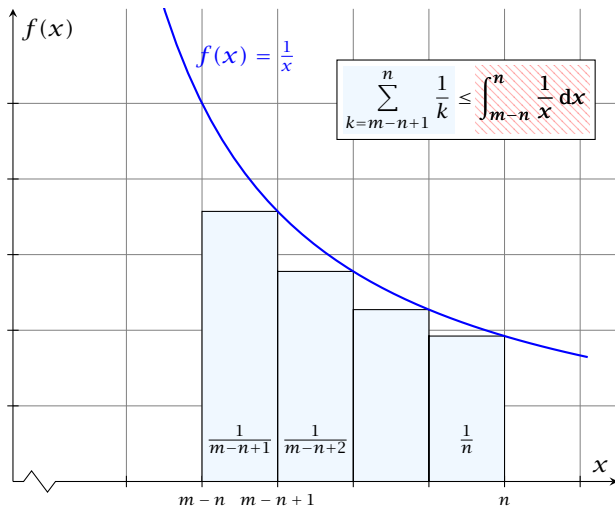
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# Deletions in Hashtables

## How do we delete in a hash-table?

- ▶ For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.
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# Deletions in Hashtables

- ▶ Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.
- ▶ One can delete an element by replacing it with a **deleted-marker**.
  - ▶ Deleted markers interrupt the probe sequence of other keys which then cannot be found anymore.
  - ▶ Deleted markers can be ignored during insertions.
  - ▶ Deleted markers can be removed during deletions.
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# Deletions for Linear Probing

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## Deletions for Linear Probing

### Algorithm 12 delete( $p$ )

```
1:  $T[p] \leftarrow \text{null}$ 
2:  $p \leftarrow \text{succ}(p)$ 
3: while  $T[p] \neq \text{null}$  do
4:    $y \leftarrow T[p]$ 
5:    $T[p] \leftarrow \text{null}$ 
6:    $p \leftarrow \text{succ}(p)$ 
7:   insert( $y$ )
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$p$  is the index into the table-cell that contains the object to be deleted.

Pointers into the hash-table become invalid.

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# Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

However, the assumption of uniform hashing that  $h$  is chosen randomly from all functions  $f: U \rightarrow [0, \dots, n-1]$  is clearly unrealistic as there are  $n^{|U|}$  such functions. Even writing down such a function would take  $|U| \log n$  bits.

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A class  $\mathcal{H}$  of hash-functions from the universe  $U$  into the set  $\{0, \dots, n-1\}$  is called **universal** if for all  $u_1, u_2 \in U$  with  $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} ,$$

where the probability is w. r. t. the choice of a random hash-function from set  $\mathcal{H}$ .

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- ▶ For any key  $u \in U$ , and  $t \in \{0, \dots, n-1\}$   $\Pr[h(u) = t] = \frac{1}{n}$ , i.e., a key is distributed uniformly within the hash-table.
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Let  $U := \{0, \dots, p-1\}$  for a prime  $p$ . Let  $\mathbb{Z}_p := \{0, \dots, p-1\}$ , and let  $\mathbb{Z}_p^* := \{1, \dots, p-1\}$  denote the set of invertible elements in  $\mathbb{Z}_p$ .

Define

$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

## Lemma 20

*The class*

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

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$$\triangleright ax + b \not\equiv ay + b \pmod{p}$$

if  $a \neq 0$ , then  $a^{-1}$  exists and we can multiply both sides with  $a^{-1}$ .

Multiplying with  $a^{-1}$  yields  $x \not\equiv y \pmod{p}$ .

Since  $x \neq y$ , we have  $x \not\equiv y \pmod{p}$ .

Therefore, the probability of a collision is  $1/n$ .

Q.E.D.  $\square$

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Multiplying with  $a \not\equiv 0 \pmod{p}$  gives

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$$t_y \equiv ay + b \pmod{p}$$

# Universal Hashing

- ▶ The hash-function does not generate collisions before the  $(\text{mod } n)$ -operation. Furthermore, every choice  $(a, b)$  is mapped to a different pair  $(t_x, t_y)$  with  $t_x := ax + b$  and  $t_y := ay + b$ .

This holds because we can compute  $a$  and  $b$  when given  $t_x$  and  $t_y$ :

$$t_x \equiv ax + b \pmod{p}$$

$$t_y \equiv ay + b \pmod{p}$$

$$t_x - t_y \equiv a(x - y) \pmod{p}$$

$$t_y \equiv ay + b \pmod{p}$$

$$a \equiv (t_x - t_y)(x - y)^{-1} \pmod{p}$$

$$b \equiv t_y - ay \pmod{p}$$



# Universal Hashing

There is a one-to-one correspondence between hash-functions (pairs  $(a, b)$ ,  $a \neq 0$ ) and pairs  $(t_x, t_y)$ ,  $t_x \neq t_y$ .

Therefore, we can view the first step (before the  $\text{mod } n$ -operation) as choosing a pair  $(t_x, t_y)$ ,  $t_x \neq t_y$  uniformly at random.

What happens when we do the  $\text{mod } n$  operation?

Fix a value  $t_x$ . There are  $p - 1$  possible values for choosing  $t_y$ .

From the range  $0, \dots, p - 1$  the values  $t_x, t_x + n, t_x + 2n, \dots$  map to  $t_x$  after the modulo-operation. These are at most  $\lceil p/n \rceil$  values.

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As  $t_y \neq t_x$  there are

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This happens with probability at most  $\frac{1}{n}$ .

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It is also possible to show that  $\mathcal{H}$  is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[ \begin{array}{l} t_x \bmod n = h_1 \\ t_y \bmod n = h_2 \end{array} \right]$$

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Note that the middle is the probability that  $h(x) = h_1$  and  $h(y) = h_2$ . The total number of choices for  $(t_x, t_y)$  is  $p(p-1)$ . The number of choices for  $t_x$  ( $t_y$ ) such that  $t_x \bmod n = h_1$  ( $t_y \bmod n = h_2$ ) lies between  $\lfloor \frac{p}{n} \rfloor$  and  $\lceil \frac{p}{n} \rceil$ .



# Universal Hashing

## Definition 21

Let  $d \in \mathbb{N}$ ;  $q \geq (d + 1)n$  be a prime; and let  $\bar{a} \in \{0, \dots, q - 1\}^{d+1}$ . Define for  $x \in \{0, \dots, q - 1\}$

$$h_{\bar{a}}(x) := \left( \sum_{i=0}^d a_i x^i \bmod q \right) \bmod n .$$

Let  $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0, \dots, q - 1\}^{d+1}\}$ . The class  $\mathcal{H}_n^d$  is  $(e, d + 1)$ -independent.

Note that in the previous case we had  $d = 1$  and chose  $a_d \neq 0$ .

# Universal Hashing

For the coefficients  $\bar{a} \in \{0, \dots, q-1\}^{d+1}$  let  $f_{\bar{a}}$  denote the polynomial

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Fix  $\ell \leq d + 1$ ; let  $x_1, \dots, x_\ell \in \{0, \dots, q - 1\}$  be keys, and let  $t_1, \dots, t_\ell$  denote the corresponding hash-function values.

Let  $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \dots, \ell\}\}$

Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \dots, \lfloor \frac{q}{n} \rfloor - 1\}\}}_{=: B_i}$$

In order to obtain the cardinality of  $A^\ell$  we choose our polynomial by fixing  $d + 1$  points.

We first fix the values for inputs  $x_1, \dots, x_\ell$ .

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Now, we choose  $d - \ell + 1$  other inputs and choose their value arbitrarily. We have  $q^{d-\ell+1}$  possibilities to do this.

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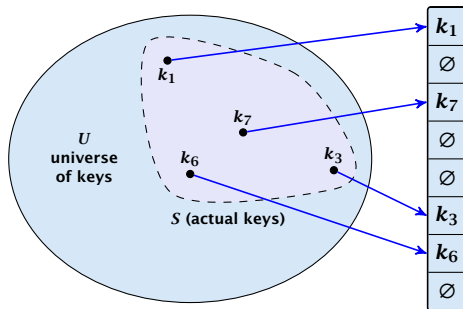
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This shows that the  $\mathcal{H}$  is  $(e, d+1)$ -universal.

The last step followed from  $q \geq (d+1)n$ , and  $\ell \leq d+1$ .

# Perfect Hashing

Suppose that we **know** the set  $S$  of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



# Perfect Hashing

Let  $m = |S|$ . We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$E[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n}.$$

If we choose  $n = m^2$  the expected number of collisions is strictly less than  $\frac{1}{2}$ .

Can we get an upper bound on the probability of having collisions?

The probability of having 1 or more collisions can be at most  $\frac{1}{2}$  as otherwise the expectation would be larger than  $\frac{1}{2}$ .

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# Perfect Hashing

We can find such a hash-function by a few trials.

However, a hash-table size of  $n = m^2$  is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from  $S$  to  $m$  buckets.

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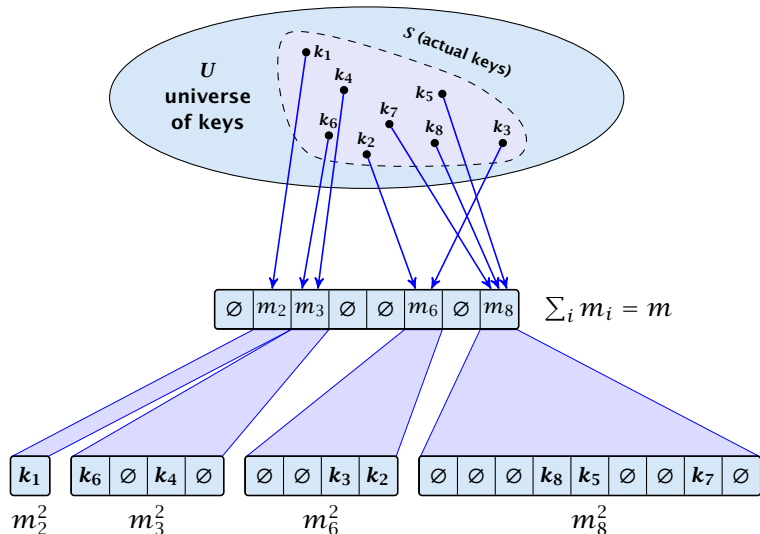
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$$= 2 \binom{m}{2} \frac{1}{m} + m = 2m - 1 .$$

# Perfect Hashing

We need only  $\mathcal{O}(m)$  time to construct a hash-function  $h$  with  $\sum_j m_j^2 = \mathcal{O}(4m)$ , because with probability at least  $1/2$  a random function from a universal family will have this property.

Then we construct a hash-table  $h_j$  for every bucket. This takes expected time  $\mathcal{O}(m_j)$  for every bucket. A random function  $h_j$  is collision-free with probability at least  $1/2$ . We need  $\mathcal{O}(m_j)$  to test this.

We only need that the hash-functions are chosen from a universal family!!!

# Cuckoo Hashing

## Goal:

Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

Two hash-tables  $T_1$  and  $T_2$  of size  $m$ , with hash functions  $h_1$  and  $h_2$ .

An object  $x$  is either stored at location  $T_1[h_1(x)]$  or  $T_2[h_2(x)]$ .

Search clearly takes constant time if the above conditions are met.



# Cuckoo Hashing

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- ▶ Two hash-tables  $T_1[0, \dots, n-1]$  and  $T_2[0, \dots, n-1]$ , with hash-functions  $h_1$ , and  $h_2$ .
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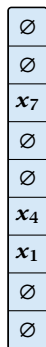
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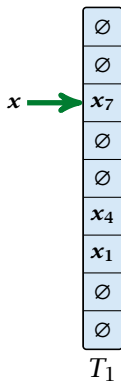
$T_1$



$T_2$

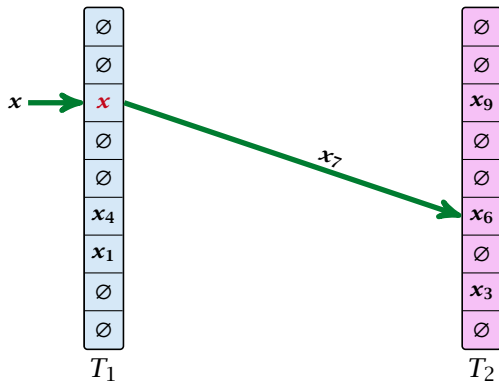
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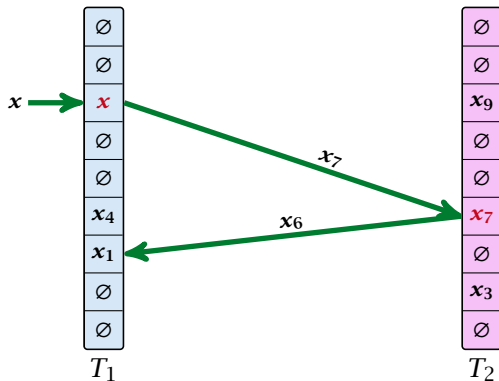
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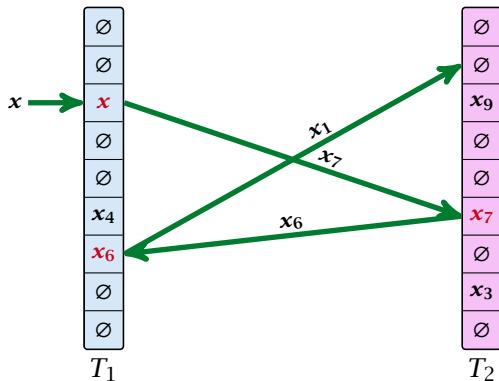
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## Algorithm 13 Cuckoo-Insert( $x$ )

```
1: if  $T_1[h_1(x)] = x \vee T_2[h_2(x)] = x$  then return  
2: steps  $\leftarrow$  1  
3: while steps  $\leq$  maxsteps do  
4:     exchange  $x$  and  $T_1[h_1(x)]$   
5:     if  $x = \text{null}$  then return  
6:     exchange  $x$  and  $T_2[h_2(x)]$   
7:     if  $x = \text{null}$  then return  
8:     steps  $\leftarrow$  steps + 1  
9: rehash() // change hash-functions; rehash everything  
10: Cuckoo-Insert( $x$ )
```

# Cuckoo Hashing

- ▶ We call one iteration through the while-loop a **step** of the algorithm.
- ▶ We call a sequence of iterations through the while-loop without the termination condition becoming true a **phase** of the algorithm.
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What is the expected time for an insert-operation?

We first analyze the probability that we end-up in an infinite loop (that is then terminated after `maxsteps` steps).

Formally what is the probability to enter an infinite loop that touches  $s$  different keys?

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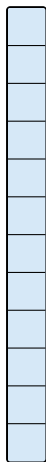


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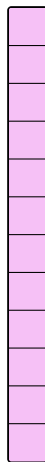
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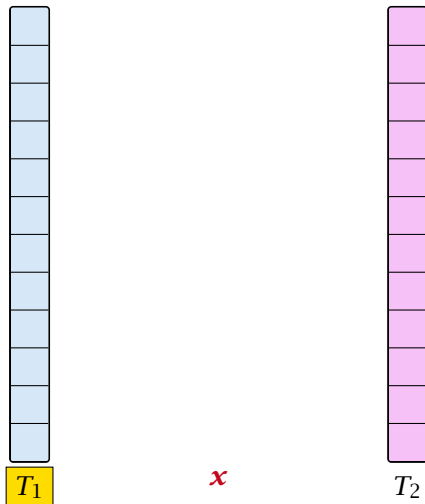


$T_1$

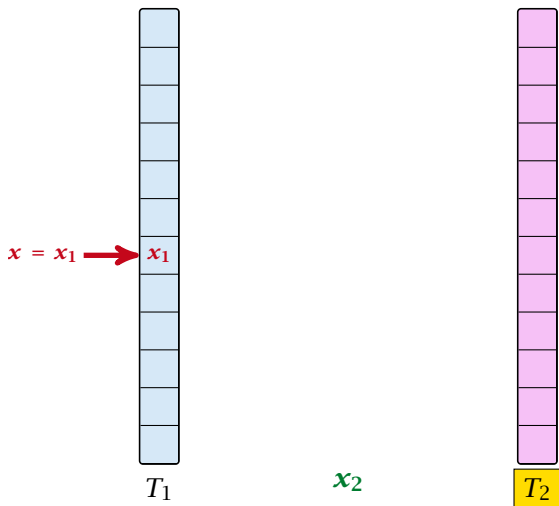


$T_2$

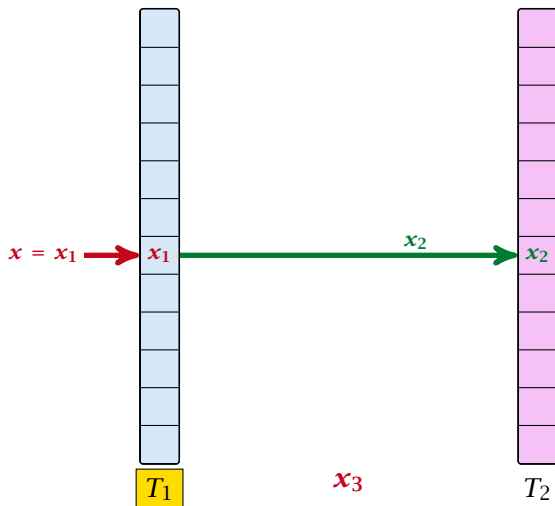
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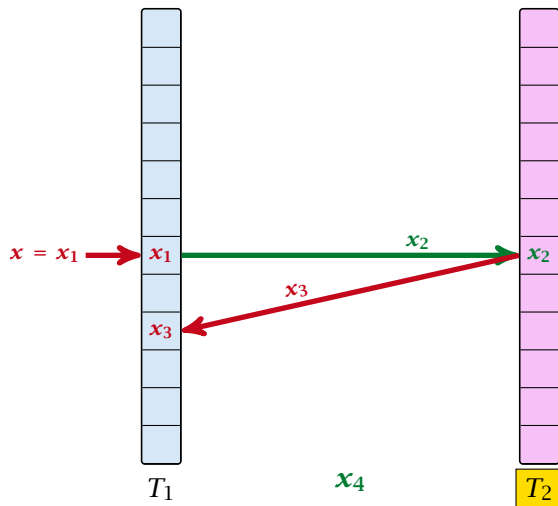
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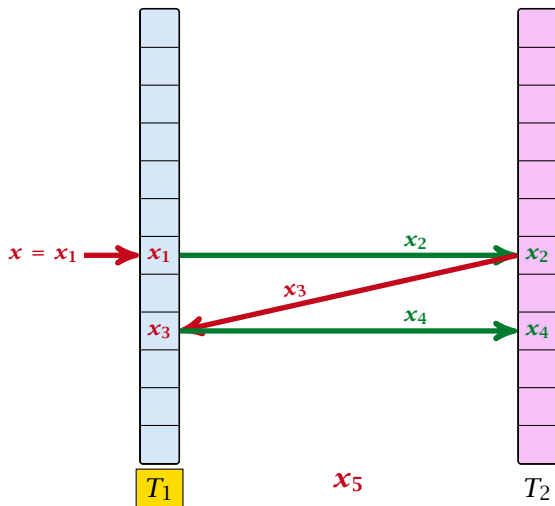
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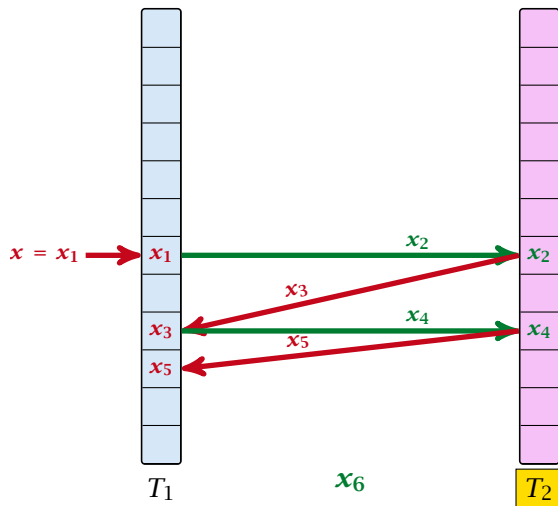
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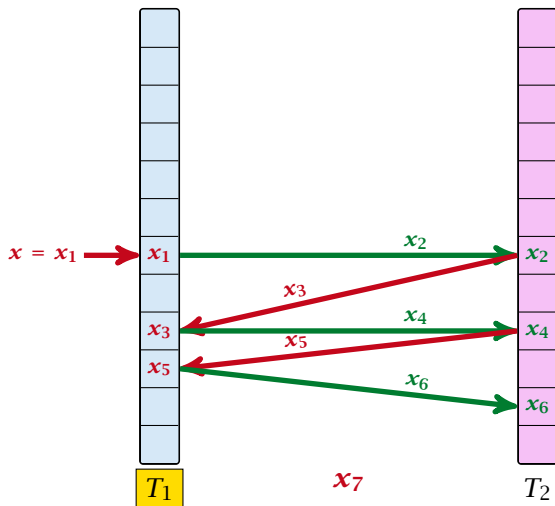


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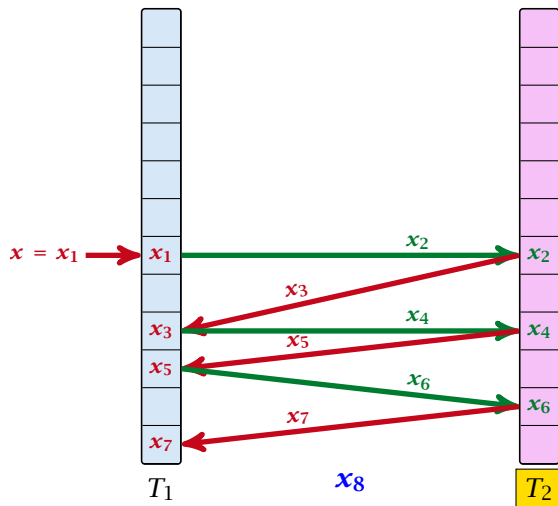




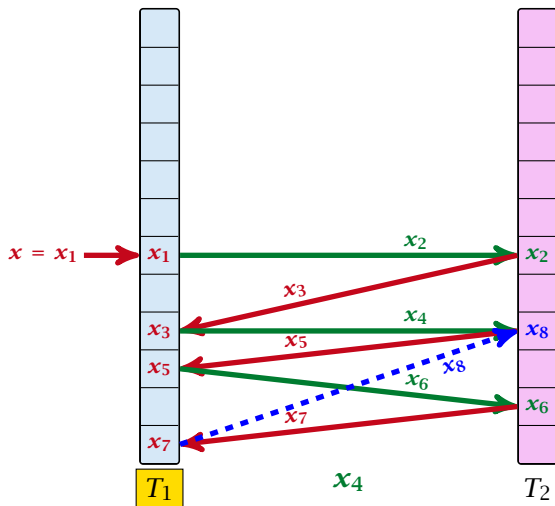
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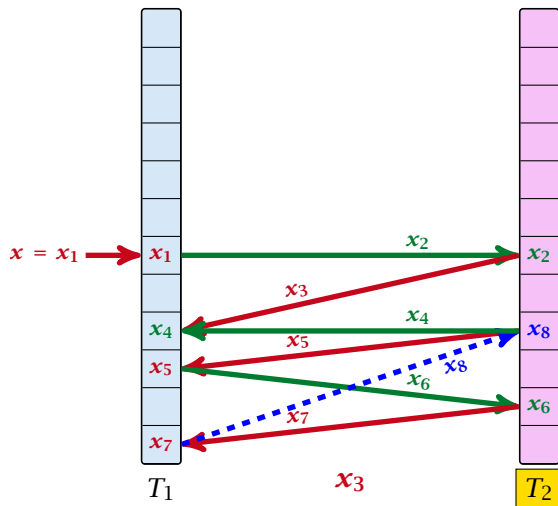
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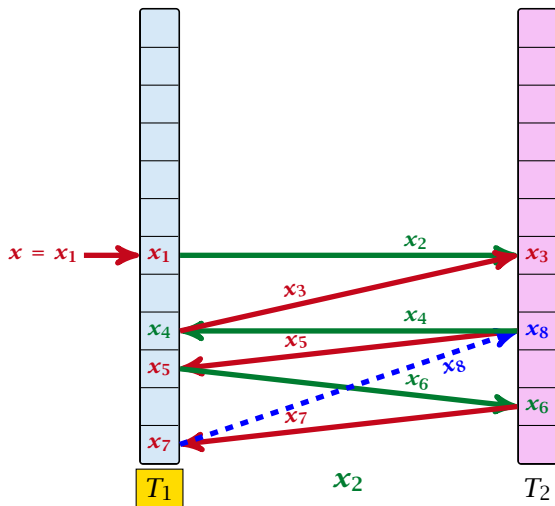
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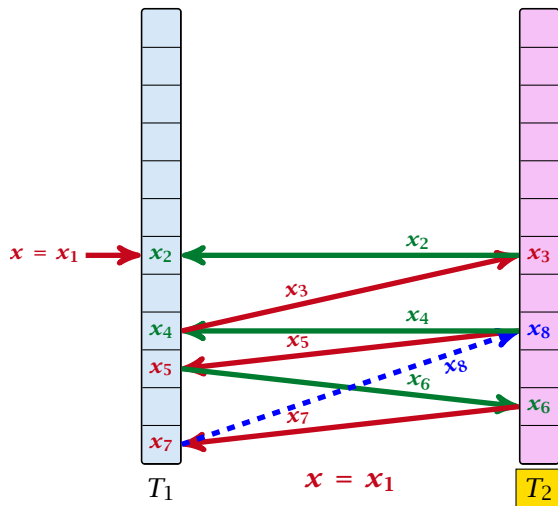
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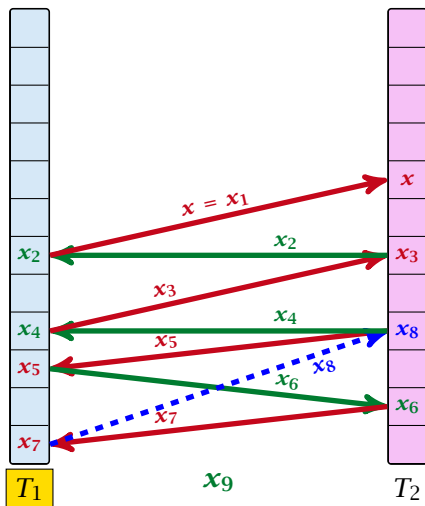
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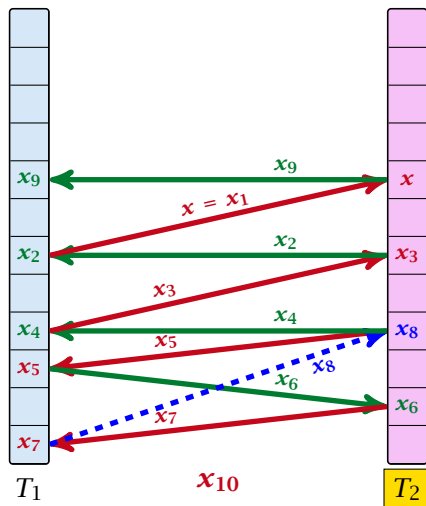
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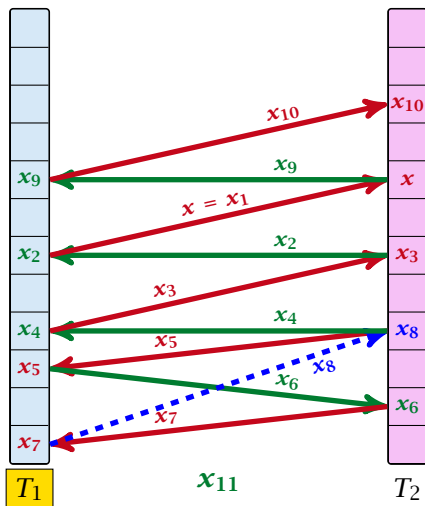


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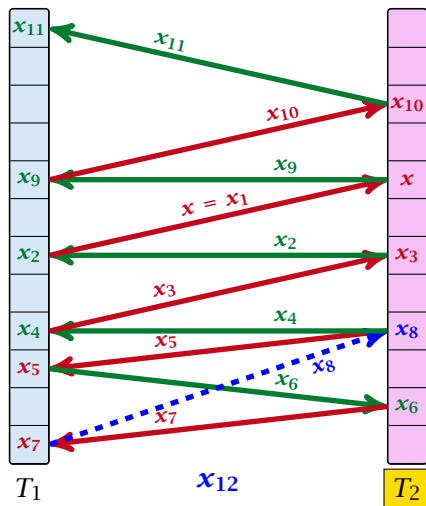




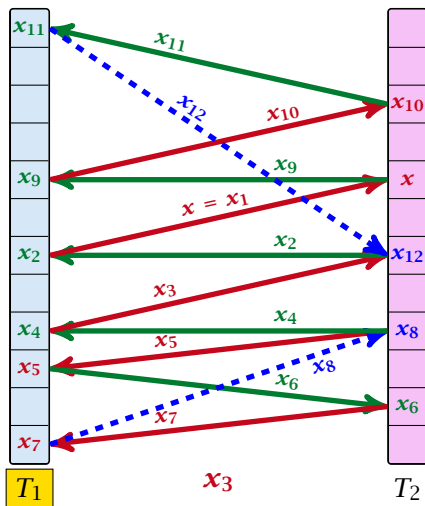
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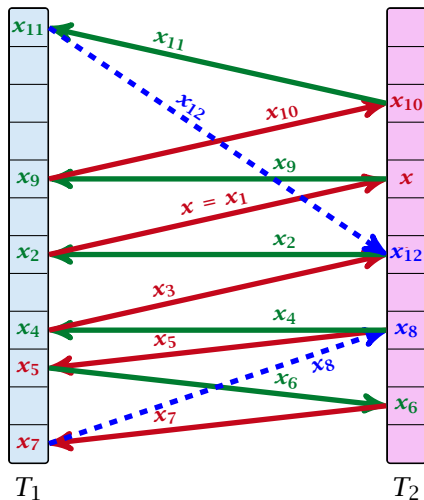
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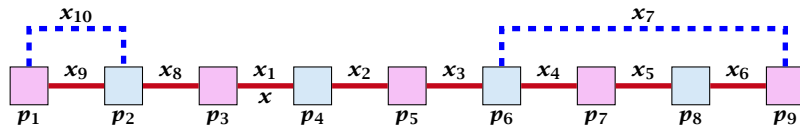
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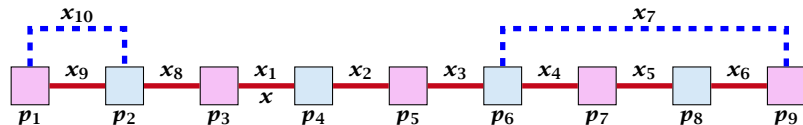


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A cycle-structure of size  $s$  is defined by

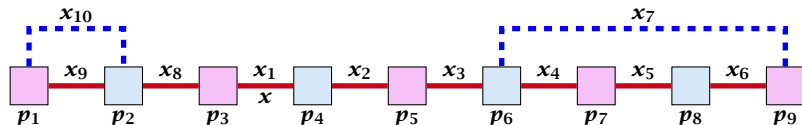
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A cycle-structure of size  $s$  is defined by

- ▶  $s - 1$  different cells (alternating btw. cells from  $T_1$  and  $T_2$ ).
- ▶  $s$  distinct keys  $x = x_1, x_2, \dots, x_s$ , linking the cells.
- ▶ The leftmost cell is “linked forward” to some cell on the right.
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- ▶ One link represents key  $x$ ; this is where the counting starts.

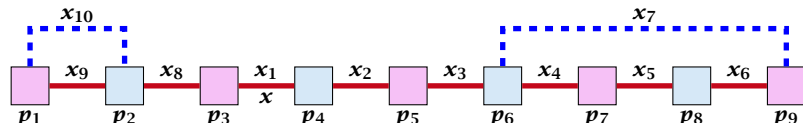
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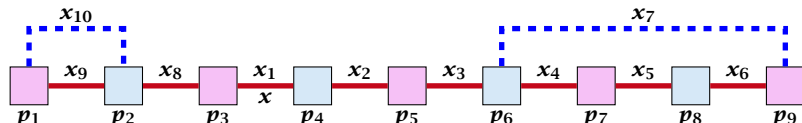


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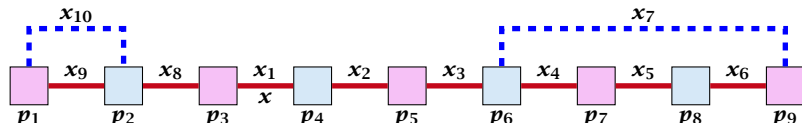
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A cycle-structure is **active** if for every key  $x_\ell$  (linking a cell  $p_i$  from  $T_1$  and a cell  $p_j$  from  $T_2$ ) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

**Observation:**

If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size  $s \geq 3$ .

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What is the probability that all keys in a cycle-structure of size  $s$  correctly map into their  $T_1$ -cell?

This probability is at most  $\frac{\mu}{n^s}$  since  $h_1$  is a  $(\mu, s)$ -independent hash-function.

What is the probability that all keys in the cycle-structure of size  $s$  correctly map into their  $T_2$ -cell?

This probability is at most  $\frac{\mu}{n^s}$  since  $h_2$  is a  $(\mu, s)$ -independent hash-function.

These events are independent.

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The probability that a given cycle-structure of size  $s$  is active is at most  $\frac{\mu^2}{n^{2s}}$ .

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The number of cycle-structures of size  $s$  is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} .$$

There are at most  $s$  possible ways to arrange the  $s$  keys in forward and backward links.

There are at most  $s$  possibilities to choose where to place

the keys in the  $s$  buckets in the forward and backward links.

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The probability that there exists an active cycle-structure is therefore at most

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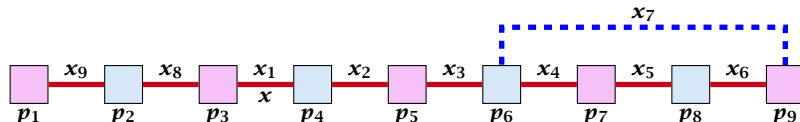
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right).$$

# Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

# Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \dots$



# Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with  $x$  in the order that they are visited during the phase.

## Lemma 22

*If the sequence is of length  $p$  then there exists a sub-sequence of at least  $\frac{p+2}{3}$  keys starting with  $x$  of distinct keys.*

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## Lemma 22

*If the sequence is of length  $p$  then there exists a sub-sequence of at least  $\frac{p+2}{3}$  keys starting with  $x$  of *distinct* keys.*

# Cuckoo Hashing

## Proof.

Let  $i$  be the number of keys (including  $x$ ) that we see before the first repeated key. Let  $j$  denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i \rightarrow x_r \rightarrow x_{r-1} \rightarrow \dots \rightarrow x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$$

As  $r \leq i - 1$  the length  $p$  of the sequence is

$$p = i + r + (j - i) \leq i + j - 1 .$$

Either sub-sequence  $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_i$  or sub-sequence  $x_1 \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j$  has at least  $\frac{p+2}{3}$  elements. □

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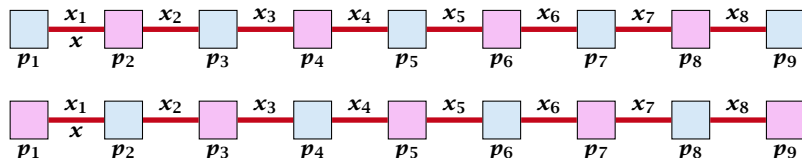
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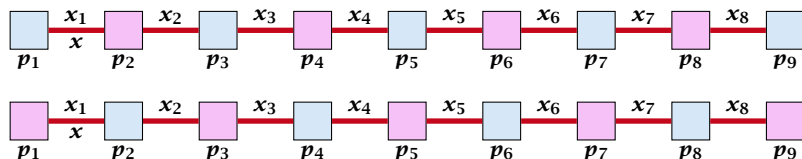
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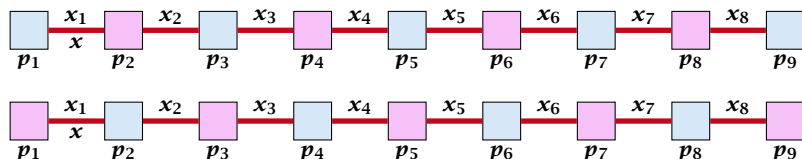
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- ▶  $s$  distinct keys  $x = x_1, x_2, \dots, x_s$ , linking the cells.
- ▶ The leftmost cell is either from  $T_1$  or  $T_2$ .

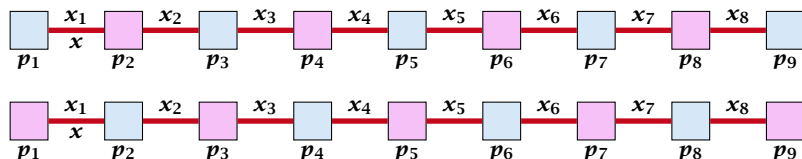
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# Cuckoo Hashing

A path-structure is **active** if for every key  $x_\ell$  (linking a cell  $p_i$  from  $T_1$  and a cell  $p_j$  from  $T_2$ ) we have

$$h_1(x_\ell) = p_i \quad \text{and} \quad h_2(x_\ell) = p_j$$

## Observation:

If a phase takes at least  $t$  steps without running into a cycle there must exist an active path-structure of size  $(2t + 2)/3$ .

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This gives  $\text{maxsteps} = \Theta(\log m)$ .

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So far we estimated

$$\Pr[\text{cycle}] \leq \mathcal{O}\left(\frac{1}{m^2}\right)$$

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for a suitable constant  $c > 0$ .

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Hence,

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

# Cuckoo Hashing

A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is  $p = \mathcal{O}(1/m^2)$  (probability  $\mathcal{O}(1/m^2)$  of running into a cycle and probability  $\mathcal{O}(1/m^2)$  of reaching maxsteps without running into a cycle).

A rehash try requires  $m$  insertions and takes expected constant time per insertion. It fails with probability  $p := \mathcal{O}(1/m)$ .

The expected number of unsuccessful rehashes is

$$\sum_{i \geq 1} p^i = \frac{1}{1-p} - 1 = \frac{p}{1-p} = \mathcal{O}(p).$$

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The expected cost for all rehashes is

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Note that  $Z_i$  is independent of  $X_j^s$ ,  $j \geq i$  (however, it is not independent of  $X_j^s$ ,  $j < i$ ). Hence,

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## What kind of hash-functions do we need?

Since  $\text{maxsteps}$  is  $\Theta(\log m)$  the largest size of a path-structure or cycle-structure contains just  $\Theta(\log m)$  different keys.

Therefore, it is sufficient to have  $(\mu, \Theta(\log m))$ -independent hash-functions.

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How do we make sure that  $n \geq (1 + \epsilon)m$ ?

- ▶ Let  $\alpha := 1/(1 + \epsilon)$ .
- ▶ Keep track of the number of elements in the table. When  $m \geq \alpha n$  we double  $n$  and do a complete re-hash (table-expand).
- ▶ Whenever  $m$  drops below  $\alpha n/4$  we divide  $n$  by 2 and do a rehash (table-shrink).
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## Lemma 23

*Cuckoo Hashing has an expected constant insert-time and a worst-case constant search-time.*

Note that the above lemma only holds if the fill-factor (number of keys/total number of hash-table slots) is at most  $\frac{1}{2(1+\epsilon)}$ .

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