17 Augmenting Paths for Matchings

Definitions.

- Given a matching M in a graph G, a vertex that is not incident to any edge of M is called a free vertex w. r. .t. M.
- For a matching M a path P in G is called an alternating path if edges in M alternate with edges not in M.
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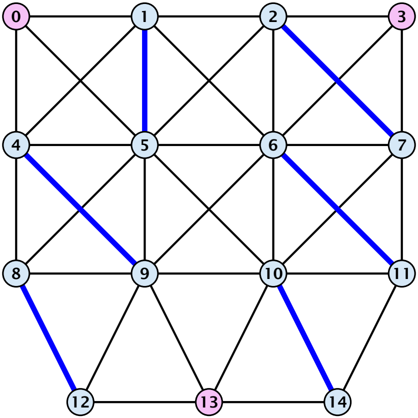
Augmenting Paths in Action



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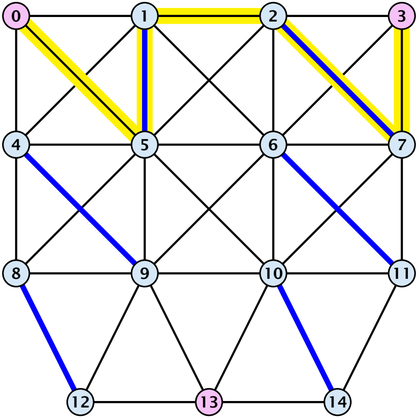

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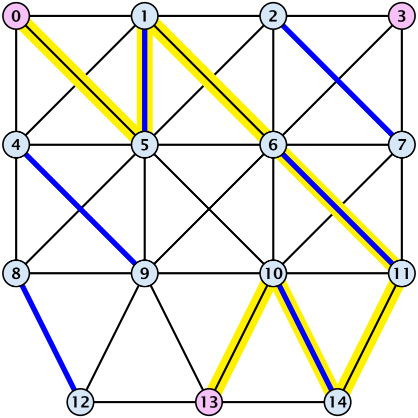

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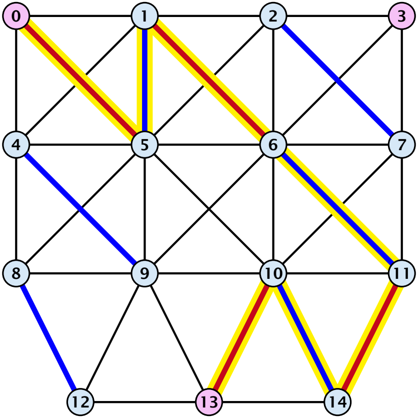

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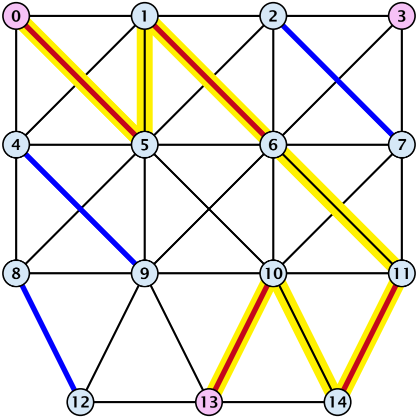

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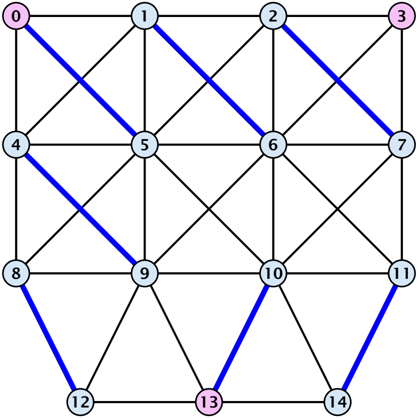

17 Augmenting Paths for Matchings

Proof.

⇒ If M is maximum there is no augmenting path P, because we could switch matching and non-matching edges along P. This gives matching $M' = M \oplus P$ with larger cardinality.

⇐ Suppose there is a matching M' with larger cardinality. Consider the graph H with edge-set $M' \oplus M$ (i.e., only edges that are in either M or M' but not in both).

Each vertex can be incident to at most two edges (one from M and one from M'). Hence, the connected components are alternating cycles or alternating path.

As $|M'| > |M|$ there is one connected component that is a path P for which both endpoints are incident to edges from M'. P is an alternating path.

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17 Augmenting Paths for Matchings

Algorithmic idea:

As long as you find an augmenting path augment your matching using this path. When you arrive at a matching for which no augmenting path exists you have a maximum matching.

Theorem 2

Let G be a graph, M a matching in G, and let u be a free vertex w.r.t. M. Further let P denote an augmenting path w.r.t. M and let $M' = M \oplus P$ denote the matching resulting from augmenting M with P. If there was no augmenting path starting at u in M then there is no augmenting path starting at u in M'.

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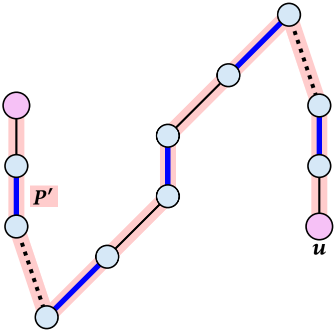

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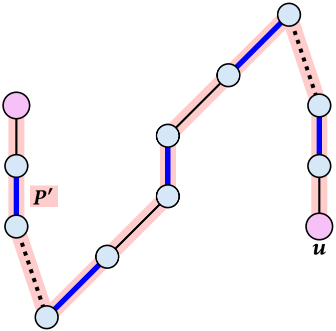

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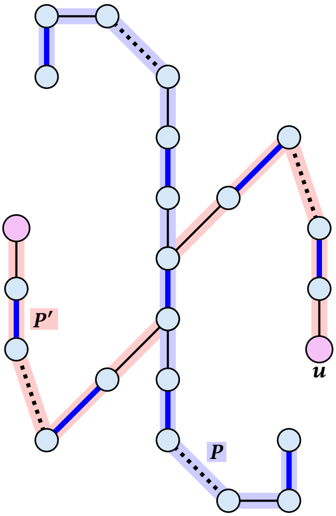

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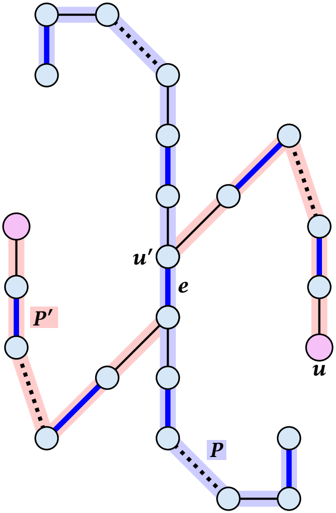

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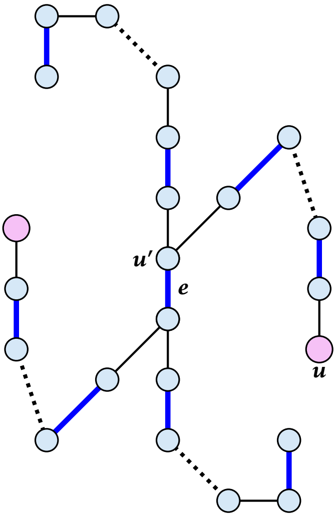

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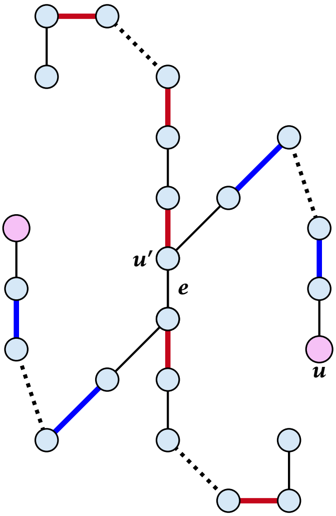

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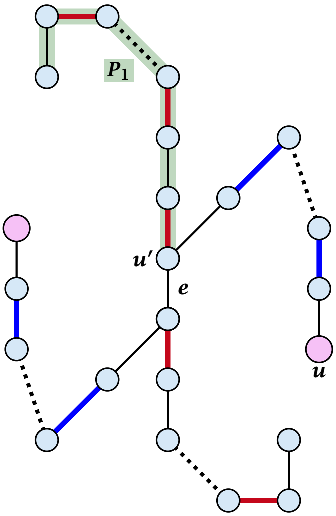

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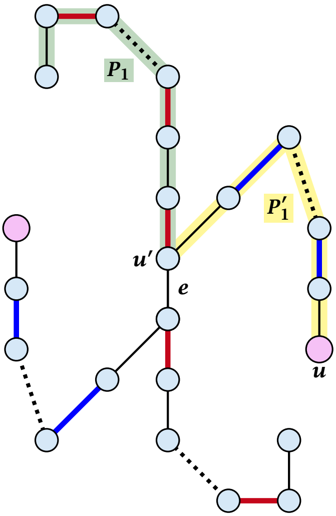

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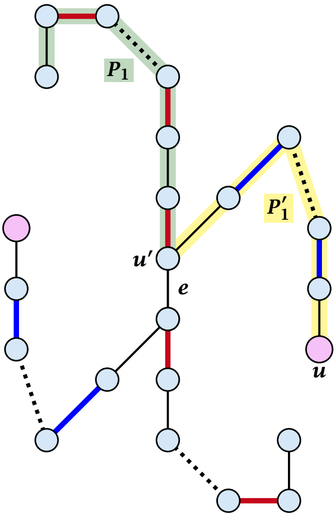
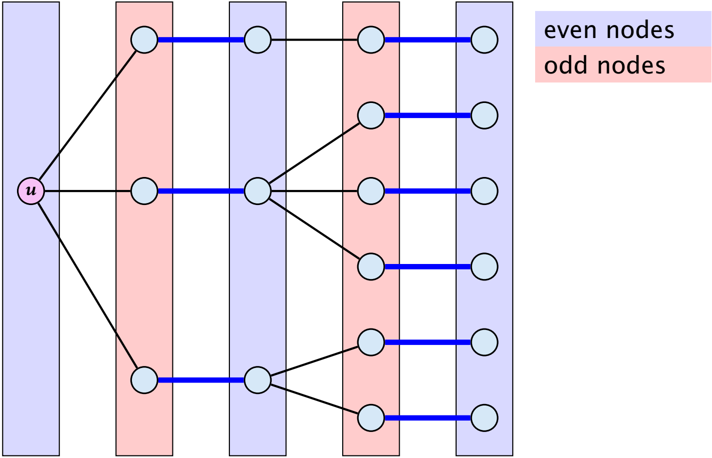
- $P_1 \circ P'_1$ is augmenting path in M (↯).



How to find an augmenting path?

Construct an alternating tree.



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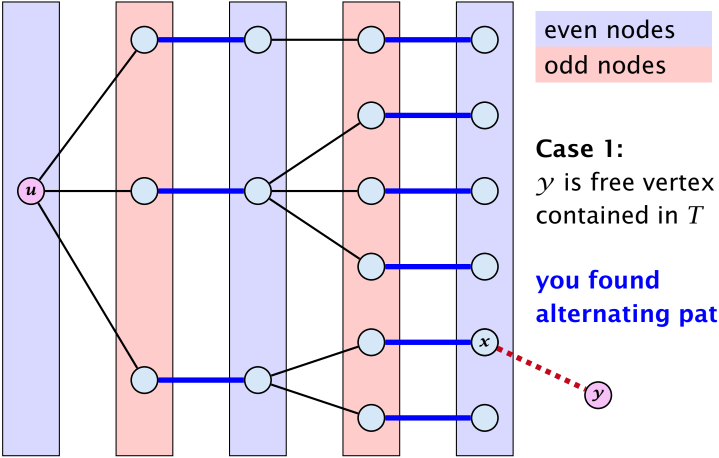
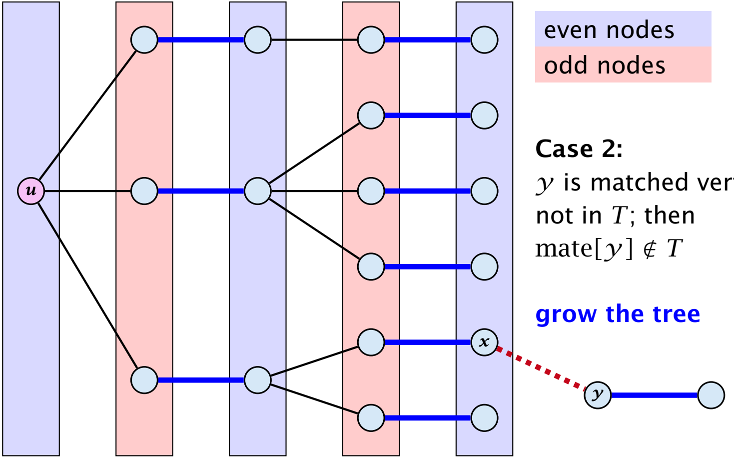


Case 1:
y is free vertex not contained in T

you found alternating path

How to find an augmenting path?

Construct an alternating tree.



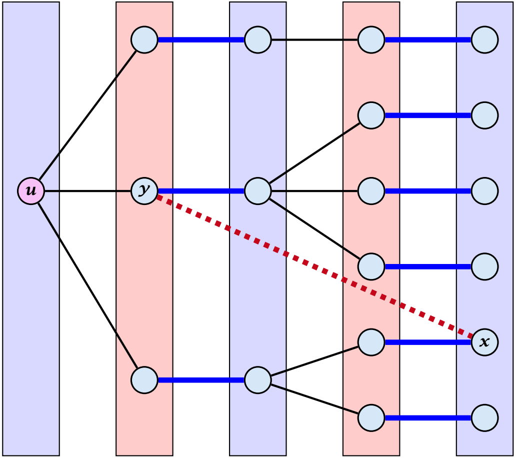
even nodes
odd nodes

Case 2:
y is matched vertex not in T; then $\text{mate}[y] \notin T$

grow the tree

How to find an augmenting path?

Construct an alternating tree.



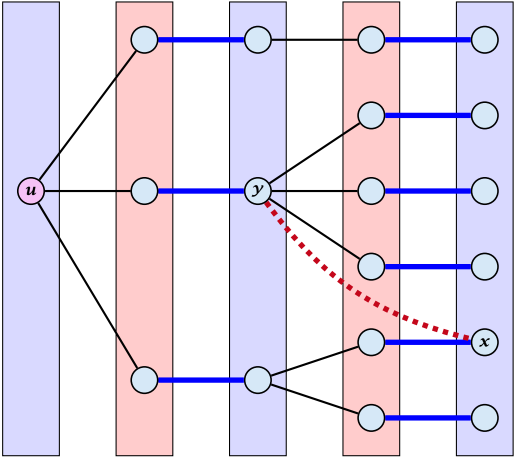
even nodes
odd nodes

Case 3:
y is already contained in T as an odd vertex

ignore successor y

How to find an augmenting path?

Construct an alternating tree.



even nodes
odd nodes

Case 4:
y is already contained in T as an even vertex

can't ignore y

does not happen in bipartite graphs

Algorithm 23 BiMatch(G, $match$)

```
for x ∈ V do mate[x] ← 0;
r ← 0; free ← n;
while free ≥ 1 and r < n do
    r ← r + 1
    if mate[r] = 0 then
        for i = 1 to n do parent[i′] ← 0
        Q ← ∅; Q.append(r); aug ← false;
        while aug = false and Q ≠ ∅ do
            x ← Q.dequeue();
            for y ∈ A_x do
                if mate[y] = 0 then
                    augm(mate, parent, y);
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graph $G = (S \cup S', E)$

$$S = \{1, \dots, n\}$$

$$S' = \{1', \dots, n'\}$$

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start with an empty matching

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free: number of unmatched nodes in S

r: root of current tree

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as long as there are unmatched nodes and we did not yet try to grow from all nodes we continue

Algorithm 23 BiMatch(G, $match$)

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r is the new node that we grow from.

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If r is free start tree construction

Algorithm 23 BiMatch(G, *match*)

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Initialize an empty tree. Note that only nodes i' have parent pointers.

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Q is a queue (BFS!!!).

aug is a Boolean that stores whether we already found an augmenting path.

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as long as we did not augment and there are still unexamined leaves continue...

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take next unexamined leaf

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if x has unmatched neighbour we found an augmenting path (note that $y \neq r$ because we are in a bipartite graph)

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do an augmentation...

Algorithm 23 BiMatch(G, $match$)

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setting aug = true ensures that the tree construction will not continue

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    if mate[r] = 0   then
        for i = 1 to n do parent[i′] ← 0
        Q ← ∅; Q.append(r); aug ← false;
        while aug = false and Q ≠ ∅   do
            x ← Q.dequeue();
            for y ∈ A_x do
                if mate[y] = 0   then
                    augm(mate, parent, y);
                    aug ← true;
                    free ← free − 1;
                else
                    if parent[y] = 0   then
                        parent[y] ← x;
                        Q.enqueue(mate[y]);
```

reduce number of free nodes

Algorithm 23 BiMatch(G, $match$)

```
for x ∈ V do mate[x] ← 0;
r ← 0; free ← n;
while free ≥ 1 and r < n do
    r ← r + 1
    if mate[r] = 0 then
        for i = 1 to n do parent[i′] ← 0
        Q ← ∅; Q.append(r); aug ← false;
        while aug = false and Q ≠ ∅ do
            x ← Q.dequeue();
            for y ∈ A_x do
                if mate[y] = 0 then
                    augm(mate, parent, y);
                    aug ← true;
                    free ← free − 1;
                else
                    if parent[y] = 0 then
                        parent[y] ← x;
                        Q.enqueue(mate[y]);
```

if y is not in the tree yet

Algorithm 23 BiMatch(G, $match$)

```
for x ∈ V do mate[x] ← 0;
r ← 0; free ← n;
while free ≥ 1 and r < n do
    r ← r + 1
    if mate[r] = 0 then
        for i = 1 to n do parent[i′] ← 0
        Q ← ∅; Q.append(r); aug ← false;
        while aug = false and Q ≠ ∅ do
            x ← Q.dequeue();
            for y ∈ A_x do
                if mate[y] = 0 then
                    augm(mate, parent, y);
                    aug ← true;
                    free ← free − 1;
                else
                    if parent[y] = 0 then
                        parent[y] ← x;
                        Q.enqueue(mate[y]);
```

...put it into the tree

Algorithm 23 BiMatch$(G, match)$

for $x \in V$ **do** $mate[x] \leftarrow 0$;
$r \leftarrow 0$; $free \leftarrow n$;
while $free \geq 1$ **and** $r < n$ **do**
    $r \leftarrow r + 1$
    if $mate[r] = 0$ **then**
        for $i = 1$ **to** n **do** $parent[i'] \leftarrow 0$
        $Q \leftarrow \emptyset$; $Q.\text{append}(r)$; $aug \leftarrow \text{false}$;
        while $aug = \text{false}$ **and** $Q \neq \emptyset$ **do**
            $x \leftarrow Q.\text{dequeue}()$;
            for $y \in A_x$ **do**
                if $mate[y] = 0$ **then**
                    augm$(mate, parent, y)$;
                    $aug \leftarrow \text{true}$;
                    $free \leftarrow free - 1$;
                else
                    if $parent[y] = 0$ **then**
                        $parent[y] \leftarrow x$;
                        $Q.\text{enqueue}(mate[y])$;

add its buddy to the set of unexamined leaves