

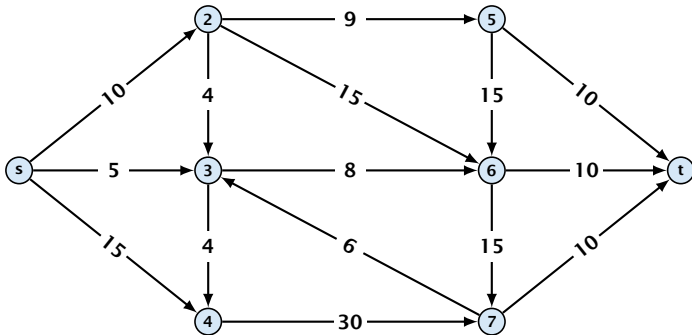
# Part IV

## Flows and Cuts

# 10 Introduction

## Flow Network

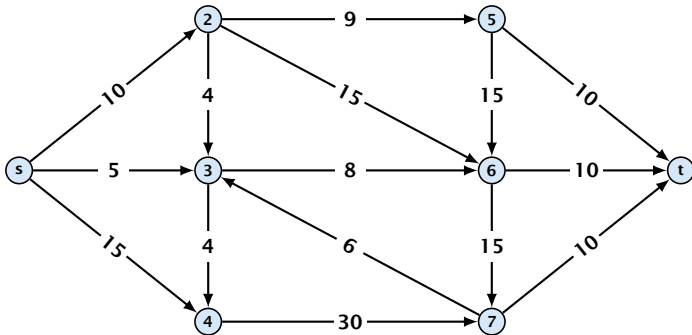
- ▶ directed graph  $G = (V, E)$ ; edge capacities  $c(e)$
- ▶ two special nodes: source  $s$ ; target  $t$ ;
- ▶ no edges entering  $s$  or leaving  $t$ ;
- ▶ at least for now: no parallel edges;



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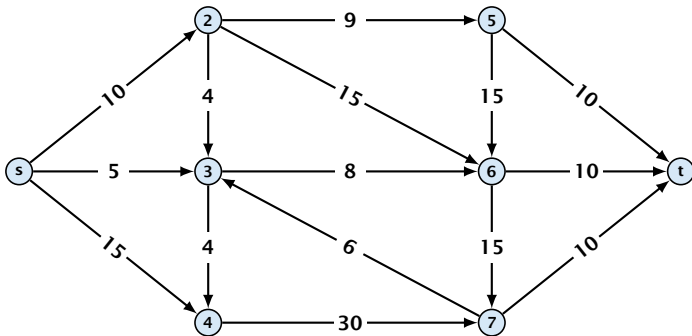
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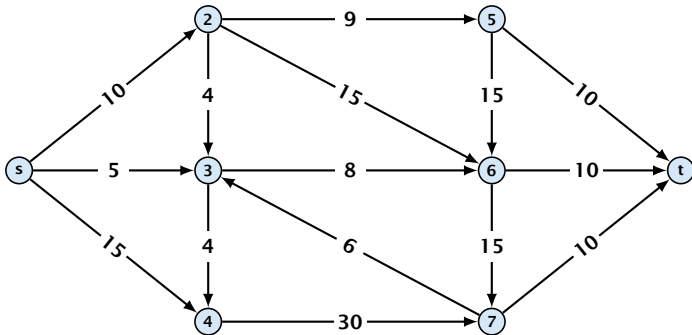
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The **capacity** of a cut  $A$  is defined as

$$\text{cap}(A, V \setminus A) := \sum_{e \in \text{out}(A)} c(e) ,$$

where  $\text{out}(A)$  denotes the set of edges of the form  $A \times V \setminus A$  (i.e. edges leaving  $A$ ).

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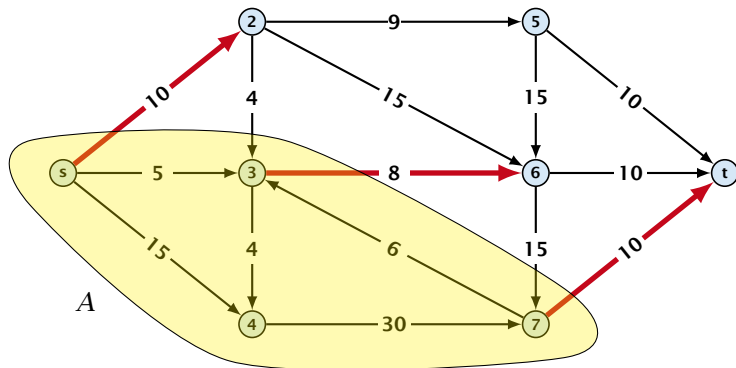
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**Minimum Cut Problem:** Find an  $(s, t)$ -cut with minimum capacity.



# Cuts

## Example 3



The capacity of the cut is  $\text{cap}(A, V \setminus A) = 28$ .

## Definition 4

An  $(s, t)$ -flow is a function  $f : E \mapsto \mathbb{R}^+$  that satisfies

1. For each edge  $e$

$$0 \leq f(e) \leq c(e) .$$

(capacity constraints)

2. For each  $v \in V \setminus \{s, t\}$

$$\sum_{e \in \text{out}(v)} f(e) = \sum_{e \in \text{into}(v)} f(e) .$$

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## Definition 5

The **value of an  $(s, t)$ -flow  $f$**  is defined as

$$\text{val}(f) = \sum_{e \in \text{out}(s)} f(e) .$$

**Maximum Flow Problem:** Find an  $(s, t)$ -flow with maximum value.

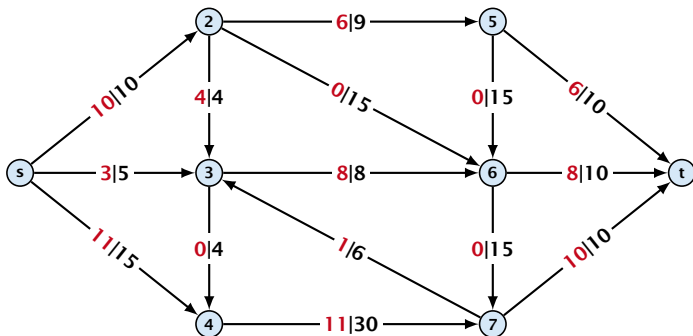
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## Example 6



The value of the flow is  $\text{val}(f) = 24$ .

## Lemma 7 (Flow value lemma)

Let  $f$  be a flow, and let  $A \subseteq V$  be an  $(s, t)$ -cut. Then the *net-flow* across the cut is equal to the amount of flow leaving  $s$ , i.e.,

$$\text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) .$$

**Proof.**

$\text{val}(f)$



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## Proof.

$$\begin{aligned}\text{val}(f) &= \sum_{e \in \text{out}(s)} f(e) \\ &= \sum_{e \in \text{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \left( \sum_{e \in \text{out}(v)} f(e) - \sum_{e \in \text{in}(v)} f(e) \right)\end{aligned}$$

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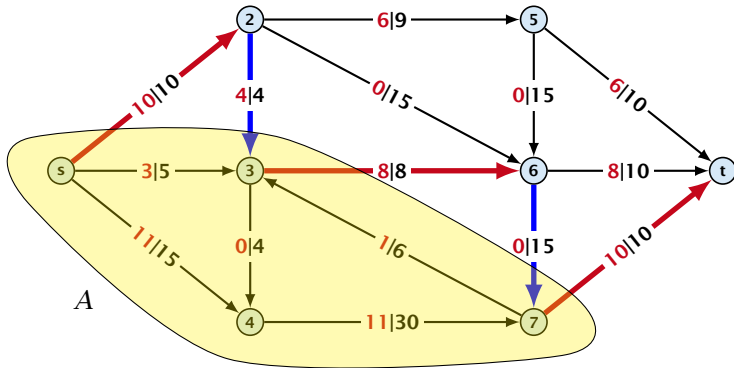
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The last equality holds since every edge with both end-points in  $A$  contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering  $A$ . □

## Example 8



## Corollary 9

Let  $f$  be an  $(s, t)$ -flow and let  $A$  be an  $(s, t)$ -cut, such that

$$\text{val}(f) = \text{cap}(A, V \setminus A).$$

Then  $f$  is a maximum flow.

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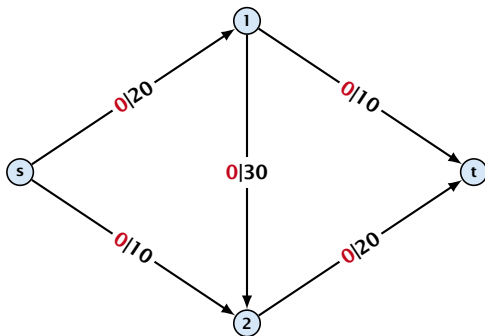
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# 11 Augmenting Path Algorithms

## Greedy-algorithm:

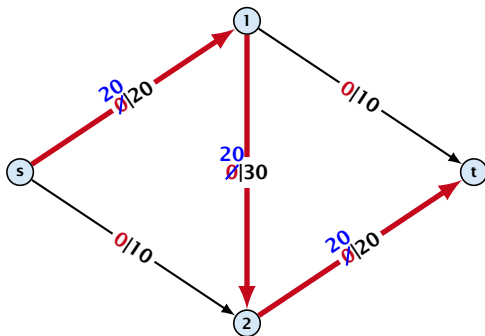
- ▶ start with  $f(e) = 0$  everywhere
- ▶ find an  $s$ - $t$  path with  $f(e) < c(e)$  on every edge
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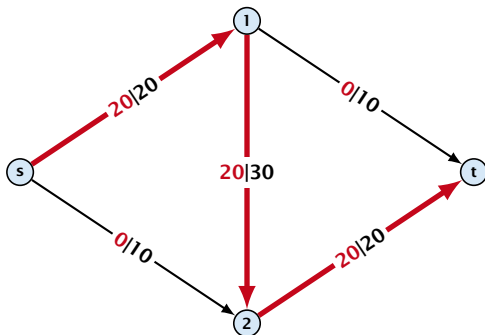
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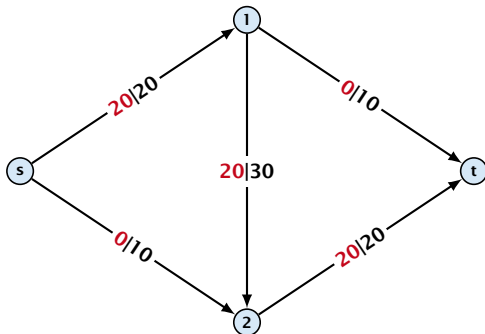




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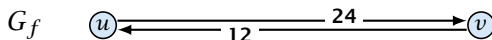
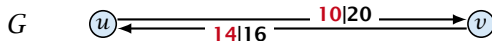
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# Augmenting Path Algorithm

## Definition 10

An **augmenting path** with respect to flow  $f$ , is a path from  $s$  to  $t$  in the auxiliary graph  $G_f$  that contains only edges with non-zero capacity.

Algorithm 1 FordFulkerson( $G = (V, E, c)$ )

- 1: Initialize  $f(e) \leftarrow 0$  for all edges.
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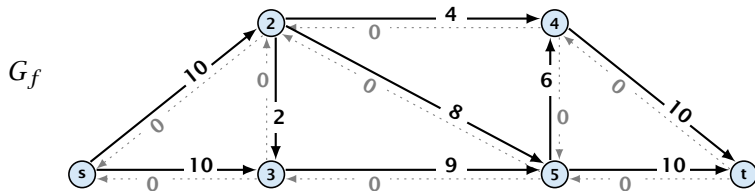
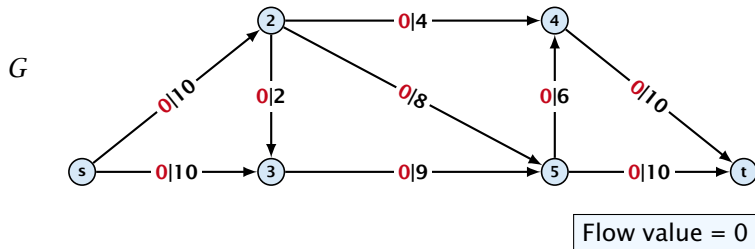
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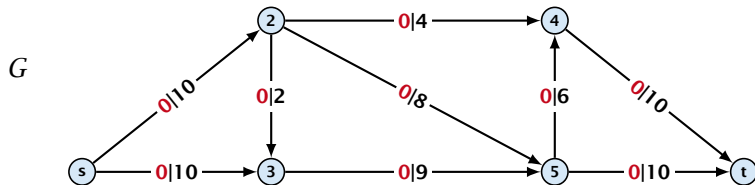
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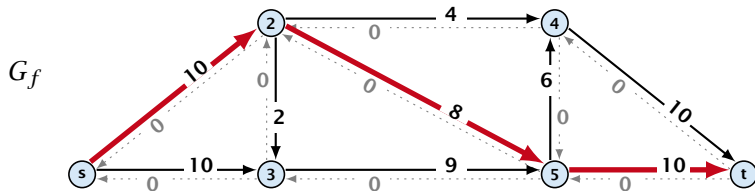




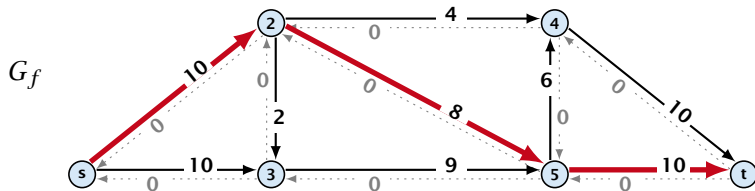
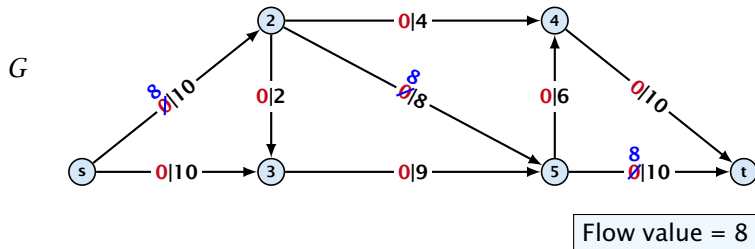
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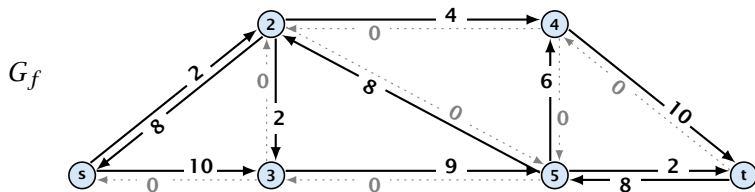
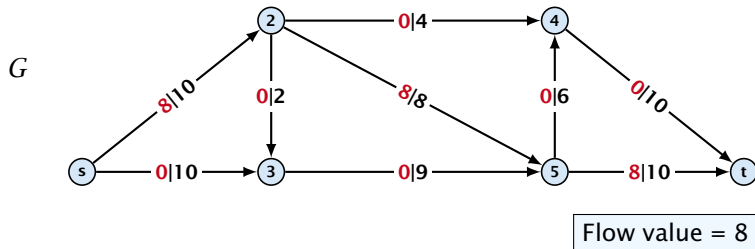
Flow value = 0



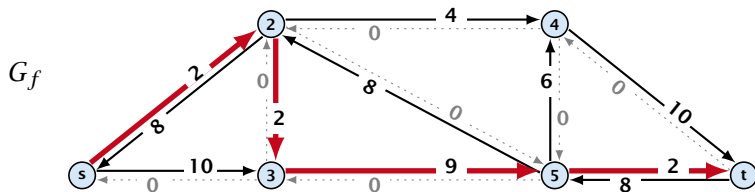
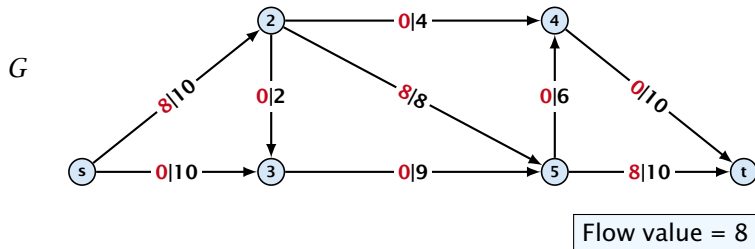
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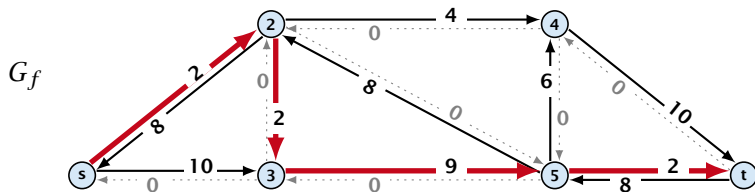
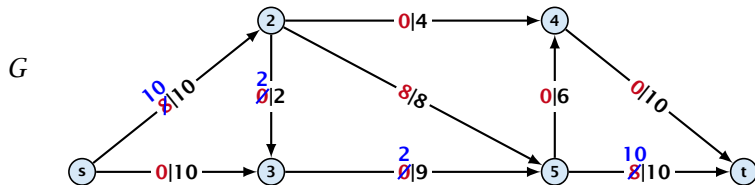
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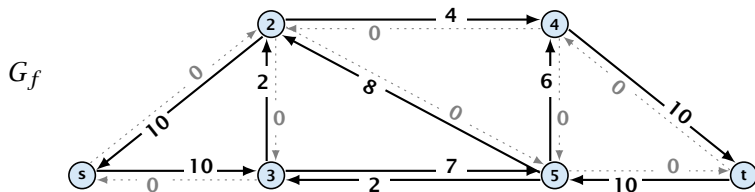
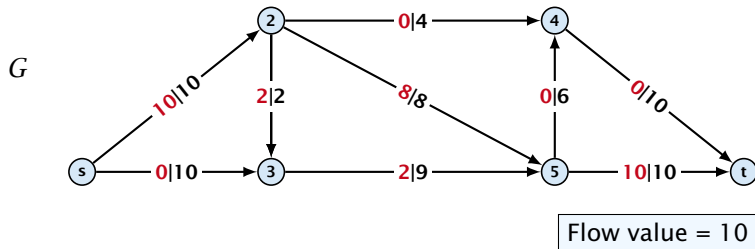
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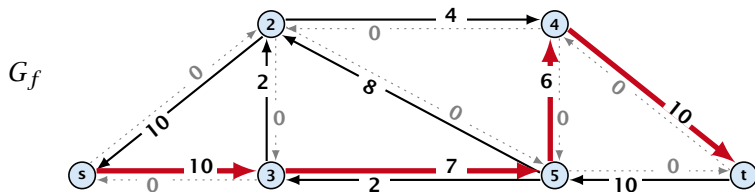
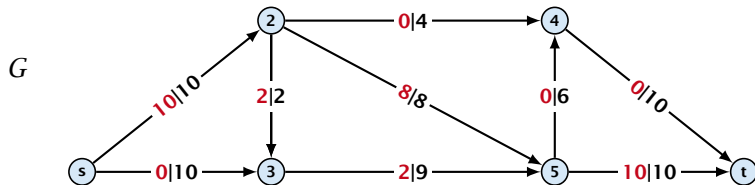
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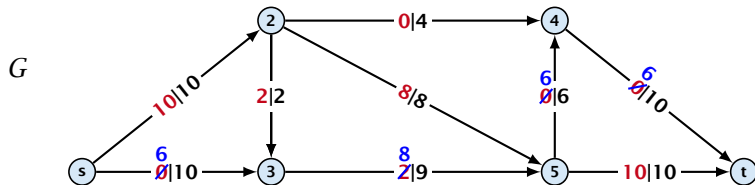
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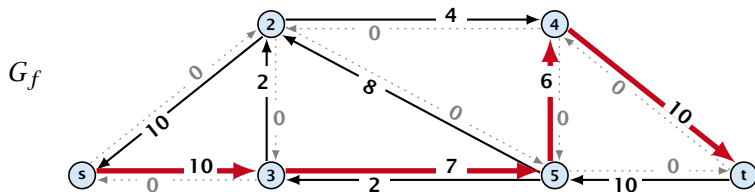
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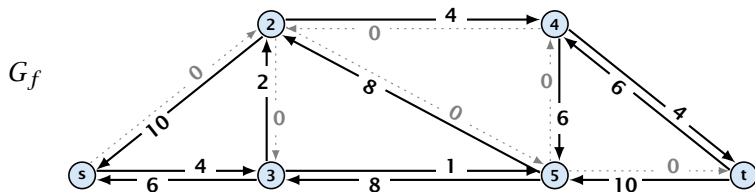
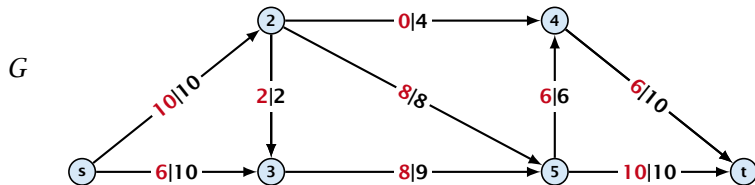


Flow value = 16

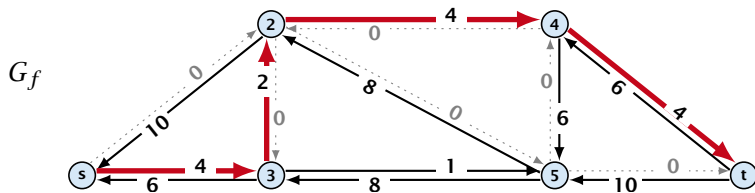
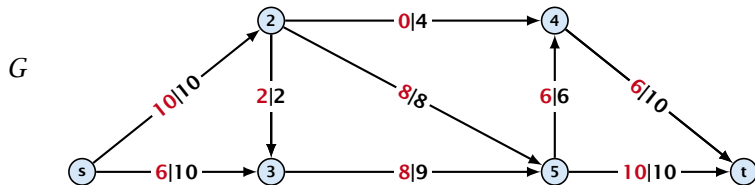




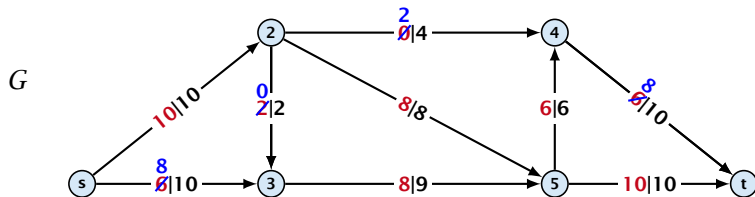
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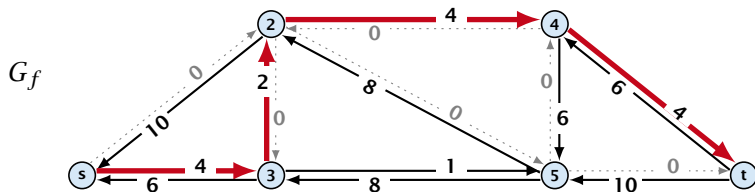
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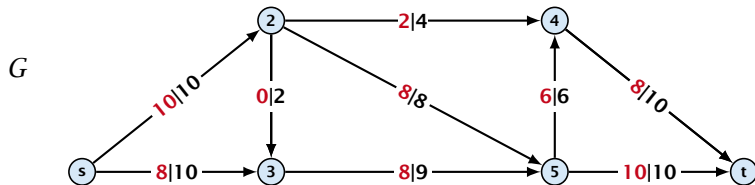
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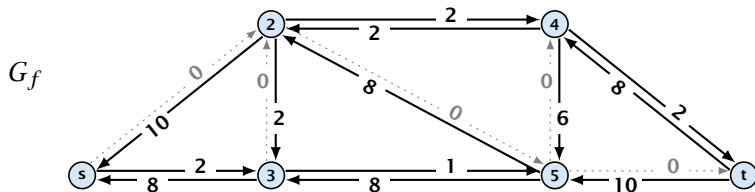
Flow value = 18



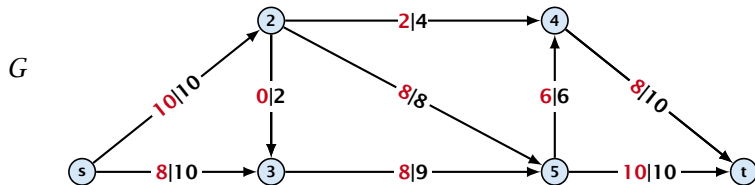
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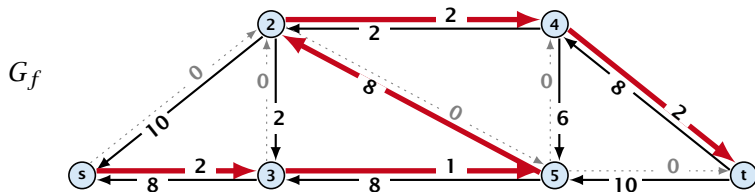
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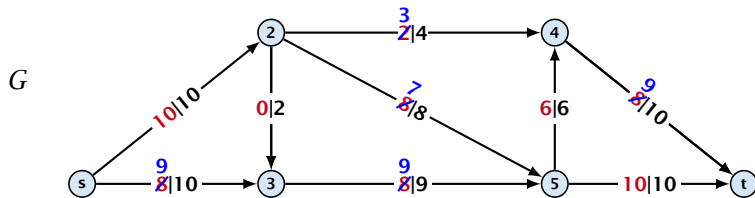
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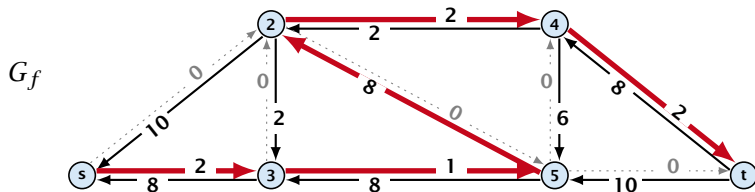
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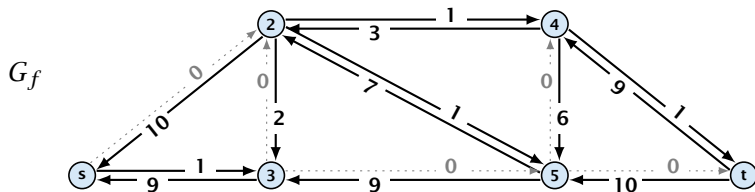
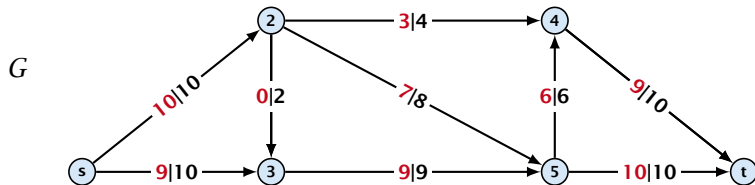
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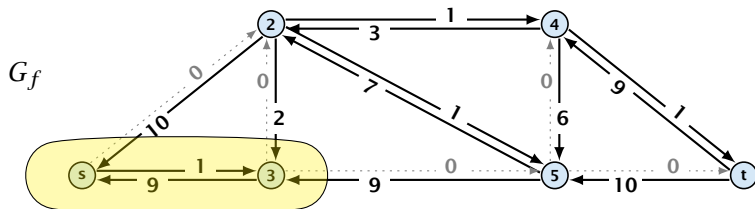
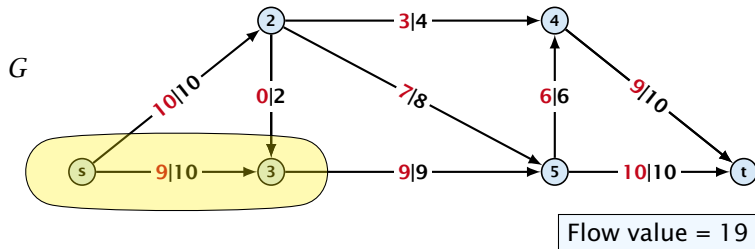
Flow value = 19



# Augmenting Path Algorithm



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# Augmenting Path Algorithm

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*The value of a maximum flow is equal to the value of a minimum cut.*

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This we already showed.

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If there were an augmenting path, we could improve the flow.  
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Let  $G_f$  be a flow with no augmenting paths.

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This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving  $A$ .

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Assumption:

All capacities are integers between 1 and  $C$ .

Invariant:

Every flow value  $f(e)$  and every residual capacity  $c_f(e)$  remains integral throughout the algorithm.

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The algorithm terminates in at most  $\text{val}(f^*) \leq nC$  iterations, where  $f^*$  denotes the maximum flow. Each iteration can be implemented in time  $\mathcal{O}(m)$ . This gives a total running time of  $\mathcal{O}(nmC)$ .

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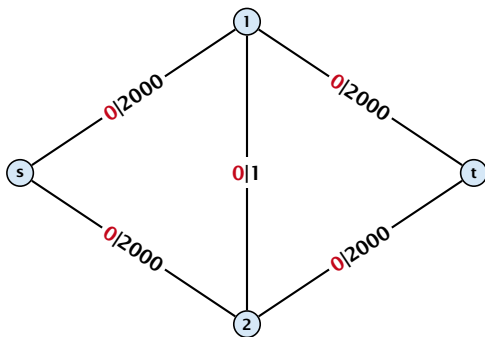
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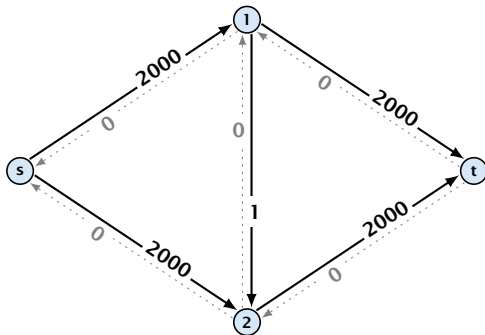
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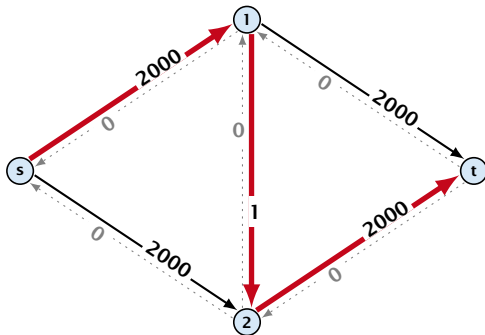


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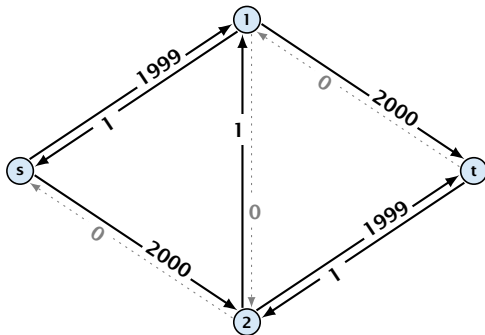
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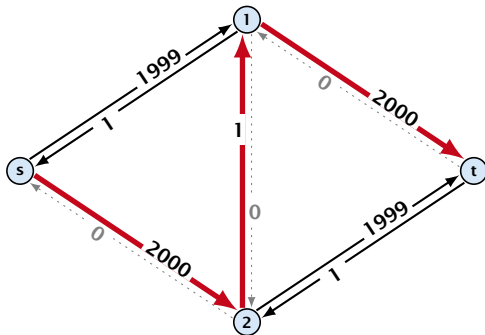


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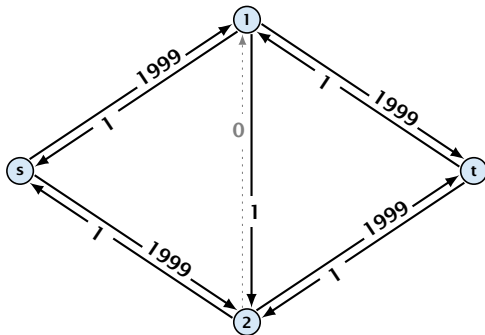


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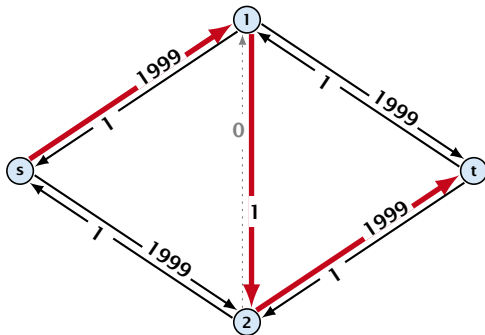


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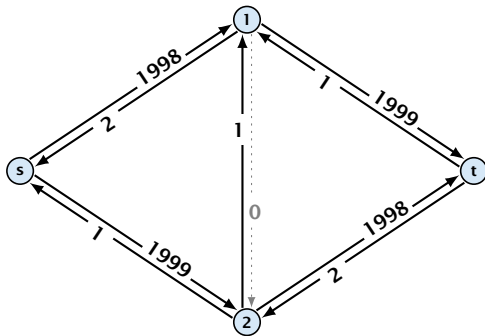


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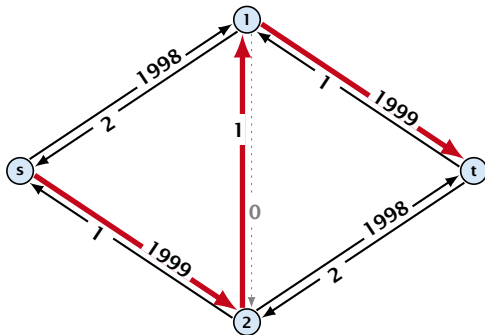


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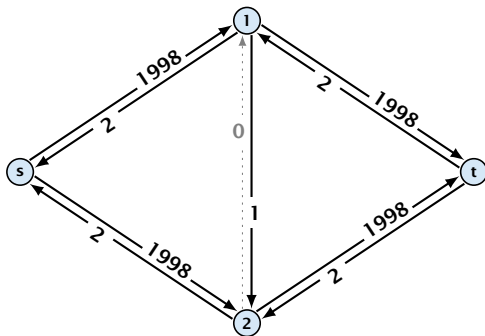


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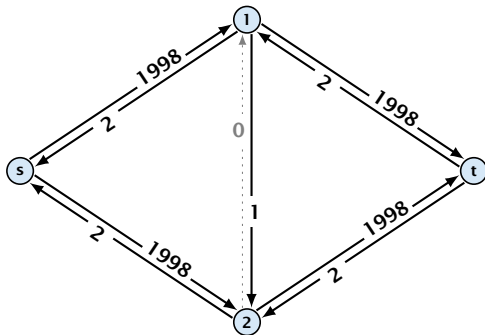


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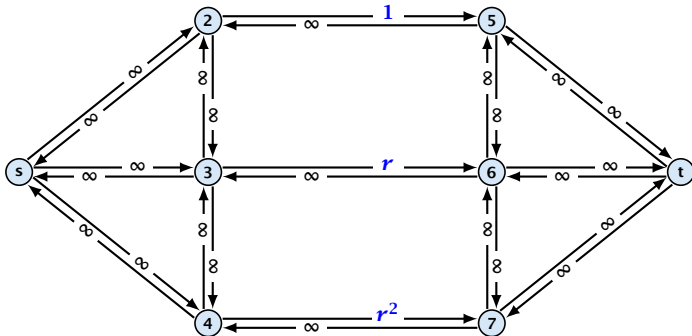
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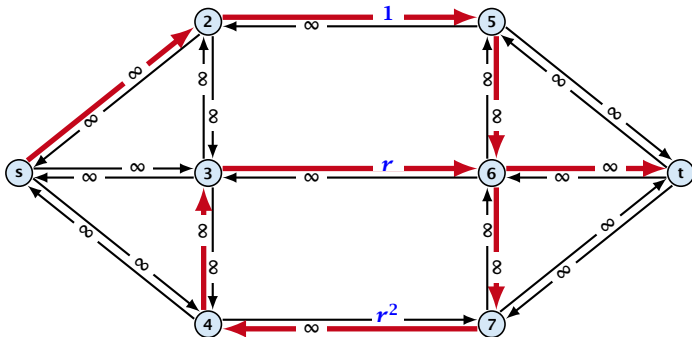
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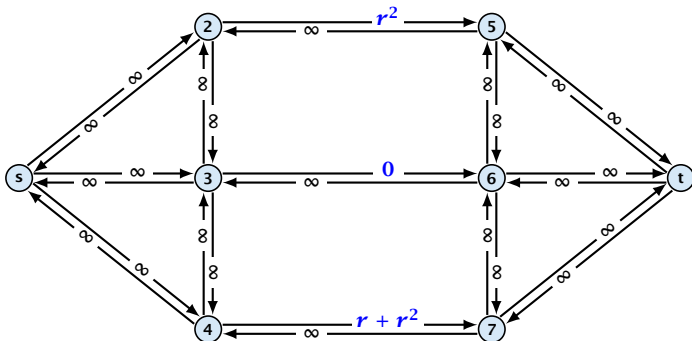
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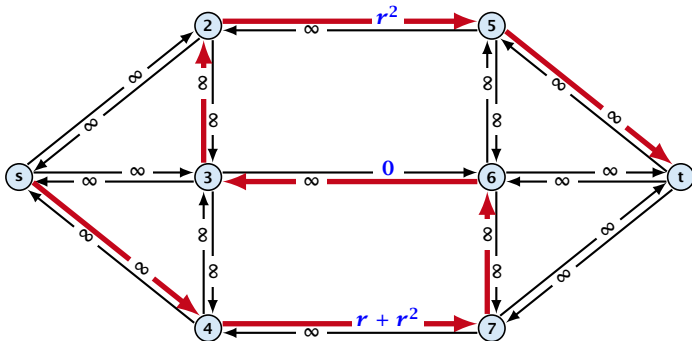
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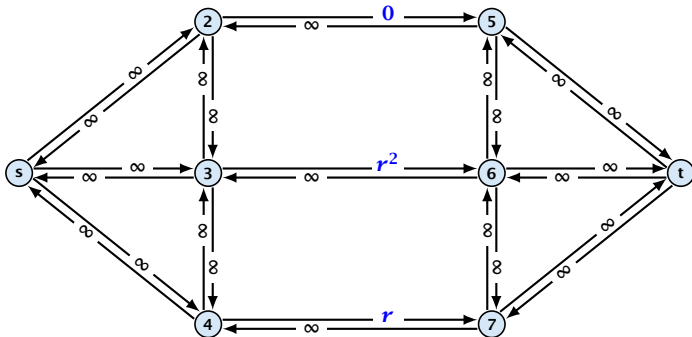
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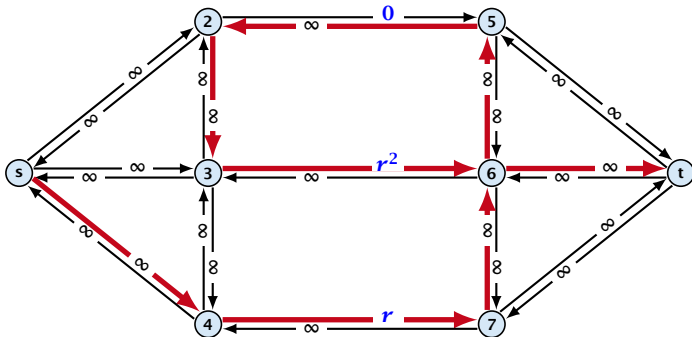
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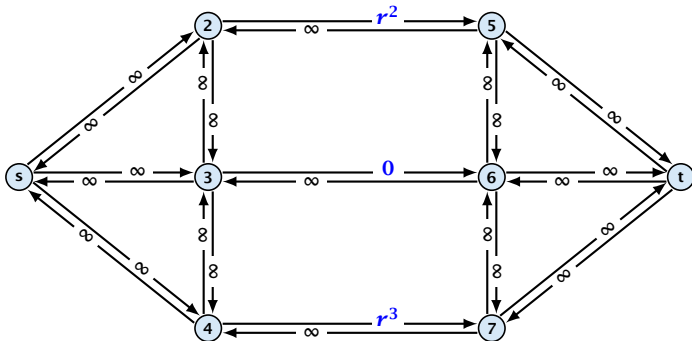
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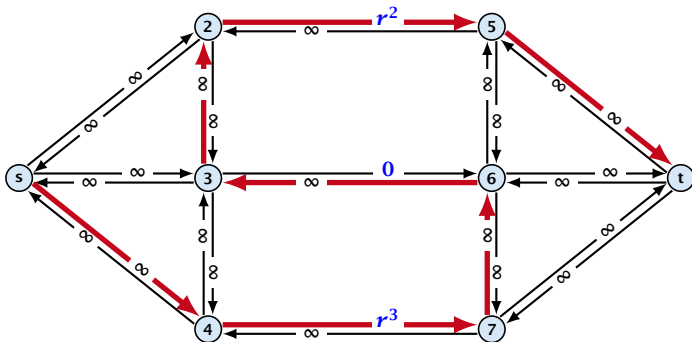
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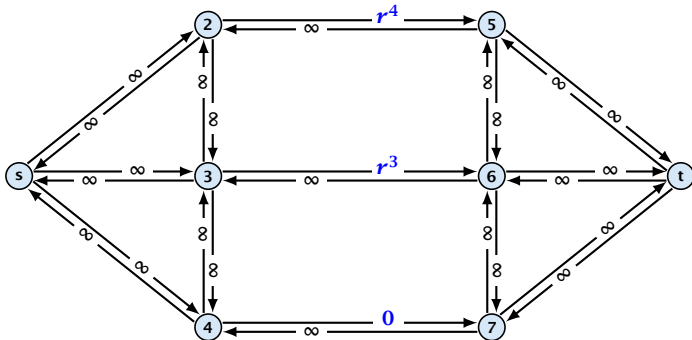
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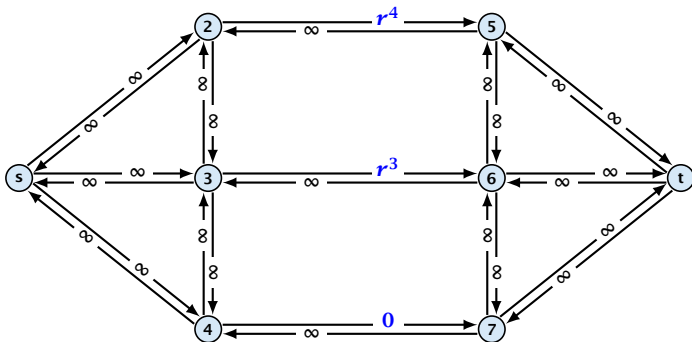
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## Lemma 15

*The length of the shortest augmenting path never decreases.*

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We can find the shortest augmenting paths in time  $\mathcal{O}(m)$ .

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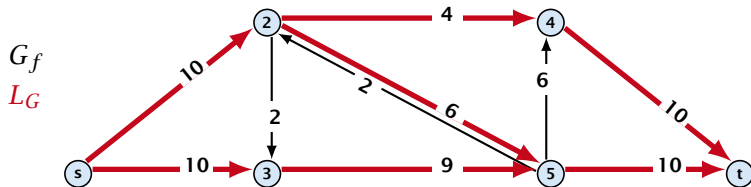
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In the following we assume that the residual graph  $G_f$  does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

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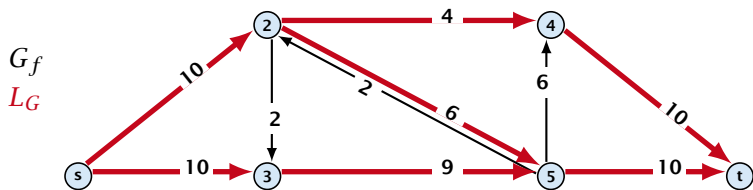
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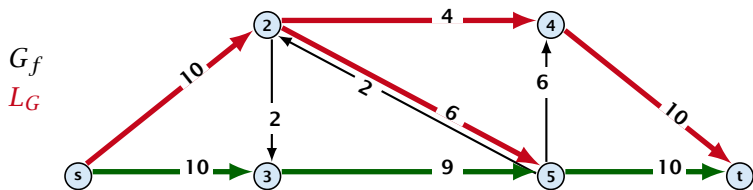
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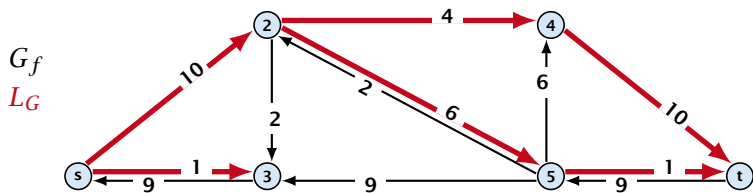
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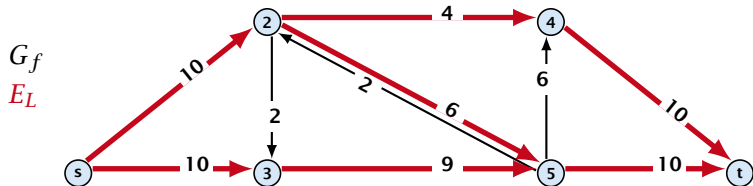
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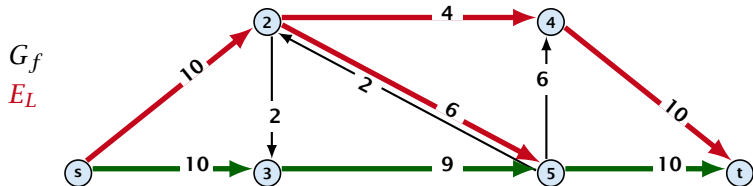
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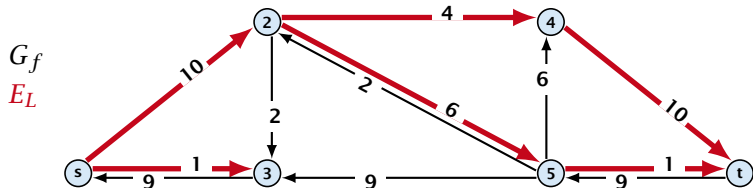
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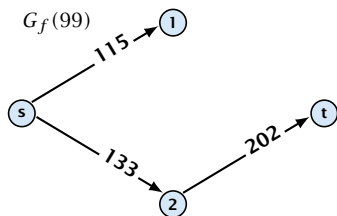
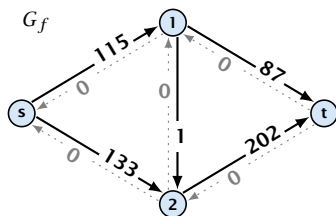
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## Algorithm 2 maxflow( $G, s, t, c$ )

```
1: foreach  $e \in E$  do  $f_e \leftarrow 0$ ;  
2:  $\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$   
3: while  $\Delta \geq 1$  do  
4:    $G_f(\Delta) \leftarrow \Delta$ -residual graph  
5:   while there is augmenting path  $P$  in  $G_f(\Delta)$  do  
6:      $f \leftarrow \text{augment}(f, c, P)$   
7:      $\text{update}(G_f(\Delta))$   
8:    $\Delta \leftarrow \Delta/2$   
9: return  $f$ 
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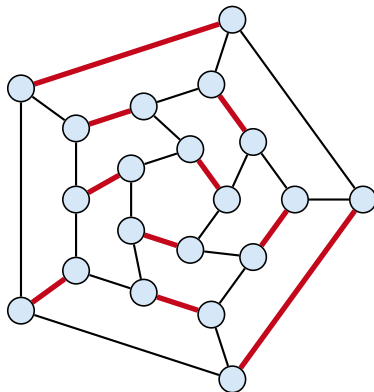
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## Theorem 23

*We need  $\mathcal{O}(m \log C)$  augmentations. The algorithm can be implemented in time  $\mathcal{O}(m^2 \log C)$ .*

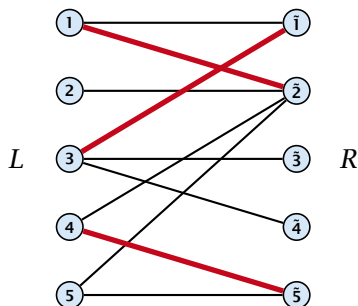
# Matching

- ▶ Input: undirected graph  $G = (V, E)$ .
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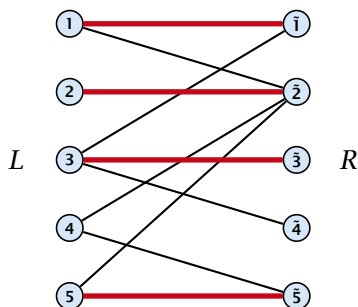
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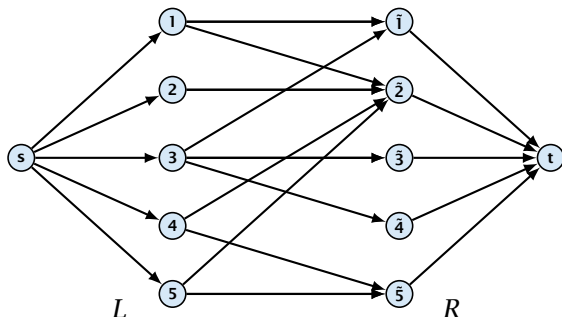
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# Maxflow Formulation

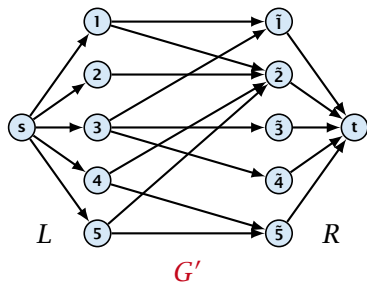
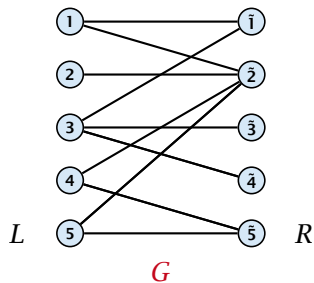
- ▶ Input: undirected, bipartite graph  $G = (L \uplus R \uplus \{s, t\}, E')$ .
- ▶ Direct all edges from  $L$  to  $R$ .
- ▶ Add source  $s$  and connect it to all nodes on the left.
- ▶ Add  $t$  and connect all nodes on the right to  $t$ .
- ▶ All edges have unit capacity.



# Proof

## Max cardinality matching in $G \leq$ value of maxflow in $G'$

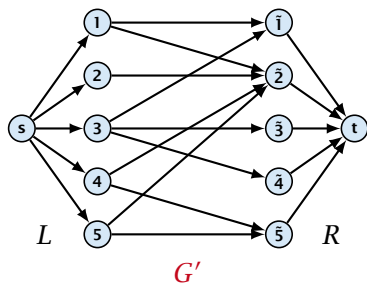
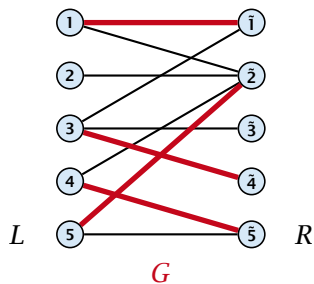
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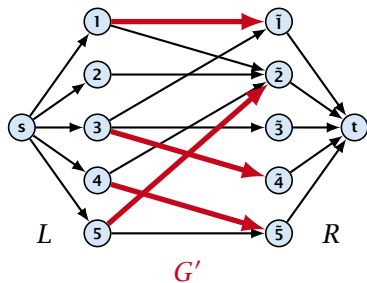
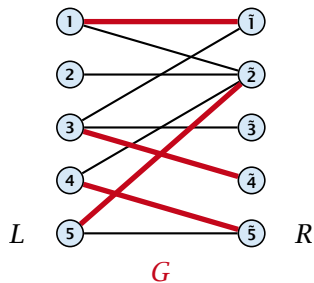
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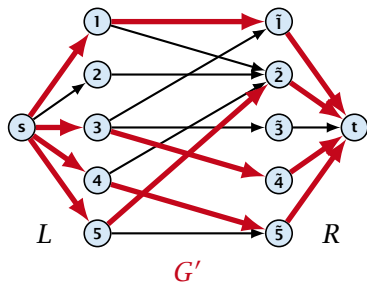
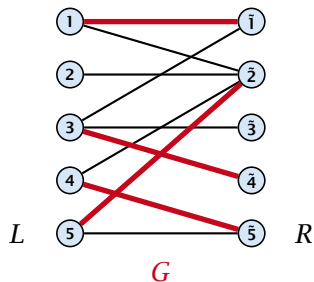
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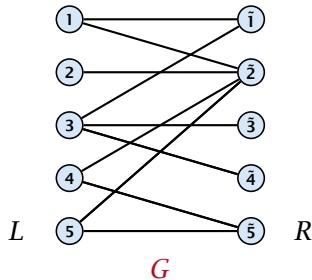
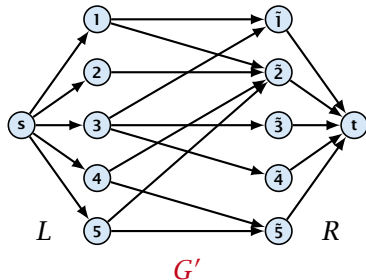
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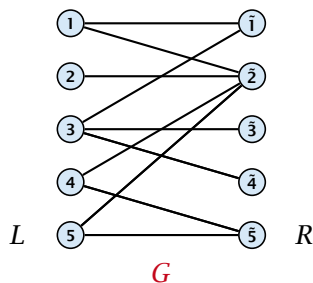
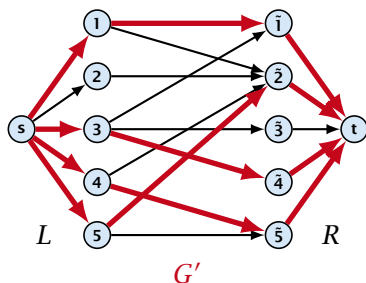
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- ▶ Consider  $M =$  set of edges from  $L$  to  $R$  with  $f(e) = 1$ .
- ▶ Each node in  $L$  and  $R$  participates in at most one edge in  $M$ .
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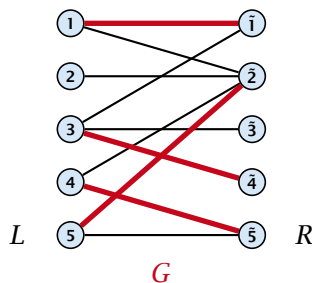
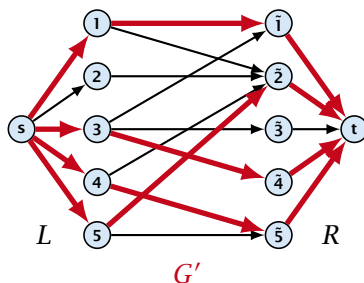
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# 12.1 Matching

## Which flow algorithm to use?

- ▶ Generic augmenting path:  $\mathcal{O}(m \text{val}(f^*)) = \mathcal{O}(mn)$ .
- ▶ Capacity scaling:  $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$ .
- ▶ Shortest augmenting path:  $\mathcal{O}(mn^2)$ .

For **unit capacity simple graphs** shortest augmenting path can be implemented in time  $\mathcal{O}(m\sqrt{n})$ .

# Baseball Elimination

<i>team</i> <i>i</i>	<i>wins</i> $w_i$	<i>losses</i> $\ell_i$	<i>remaining games</i>			
			<i>Atl</i>	<i>Phi</i>	<i>NY</i>	<i>Mon</i>
Atlanta	83	71	–	1	6	1
Philadelphia	80	79	1	–	0	2
New York	78	78	6	0	–	0
Montreal	77	82	1	2	0	–

**Which team can end the season with most wins?**

- ▶ Montreal is eliminated, since even after winning all remaining games there are only 80 wins.
- ▶ But also Philadelphia is eliminated. Why?

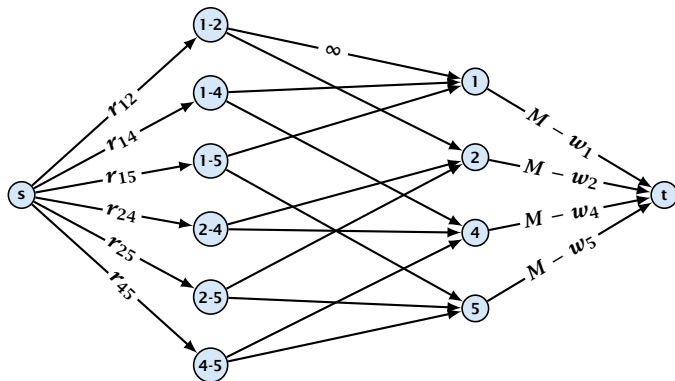
# Baseball Elimination

## Formal definition of the problem:

- ▶ Given a set  $S$  of teams, and one specific team  $z \in S$ .
- ▶ Team  $x$  has already won  $w_x$  games.
- ▶ Team  $x$  still has to play team  $y$ ,  $r_{xy}$  times.
- ▶ Does team  $z$  still have a chance to finish with the most number of wins.

# Baseball Elimination

Flow network for  $z = 3$ .  $M$  is number of wins Team 3 can still obtain.

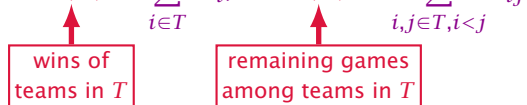


**Idea.** Distribute the results of remaining games in such a way that no team gets too many wins.

# Certificate of Elimination

Let  $T \subseteq S$  be a subset of teams. Define

$$w(T) := \sum_{i \in T} w_i, \quad r(T) := \sum_{i, j \in T, i < j} r_{ij}$$



If  $\frac{w(T)+r(T)}{|T|} > M$  then one of the teams in  $T$  will have more than  $M$  wins in the end. A team that can win at most  $M$  games is therefore eliminated.

## Theorem 24

A team  $z$  is eliminated if and only if the flow network for  $z$  does not allow a flow of value  $\sum_{i,j \in S \setminus \{z\}, i < j} r_{ij}$ .

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$$r(S \setminus \{z\}) > \text{cap}(A, V \setminus A)$$

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$$\begin{aligned} r(S \setminus \{z\}) &> \text{cap}(A, V \setminus A) \\ &\geq \sum_{i < j: i \notin T \vee j \notin T} r_{ij} + \sum_{i \in T} (M - w_i) \end{aligned}$$

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- ▶ This gives  $M < (w(T) + r(T))/|T|$ , i.e.,  $z$  is eliminated.

# Baseball Elimination

## Proof ( $\Rightarrow$ )

- ▶ Suppose we have a flow that saturates all source edges.
- ▶ We can assume that this flow is *integral*.
- ▶ For every pairing  $x$ - $y$  it defines how many games team  $x$  and team  $y$  should win.
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- ▶ This is less than  $M - w_x$  because of capacity constraints.
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# Project Selection

## Project selection problem:

- ▶ Set  $P$  of possible projects. Project  $v$  has an associated profit  $p_v$  (can be positive or negative).
- ▶ Some projects have requirements (taking course EA2 requires course EA1).
- ▶ Dependencies are modelled in a graph. Edge  $(u, v)$  means “can’t do project  $u$  without also doing project  $v$ .”
- ▶ A subset  $A$  of projects is **feasible** if the prerequisites of every project in  $A$  also belong to  $A$ .

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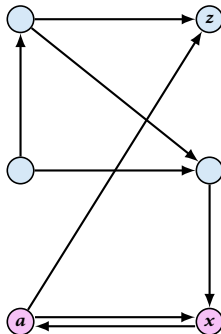
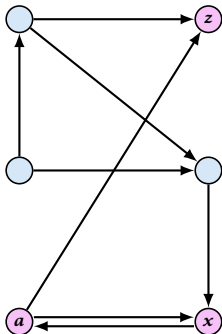
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## The prerequisite graph:

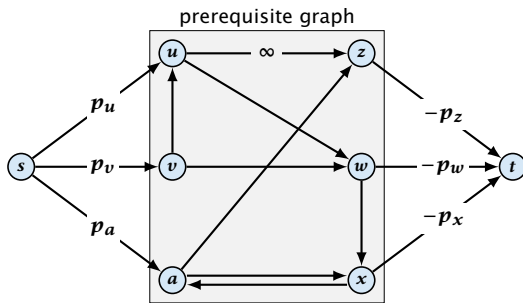
- ▶  $\{x, a, z\}$  is a feasible subset.
- ▶  $\{x, a\}$  is infeasible.



# Project Selection

## Mincut formulation:

- ▶ Edges in the prerequisite graph get infinite capacity.
- ▶ Add edge  $(s, v)$  with capacity  $p_v$  for nodes  $v$  with positive profit.
- ▶ Create edge  $(v, t)$  with capacity  $-p_v$  for nodes  $v$  with negative profit.



## Theorem 25

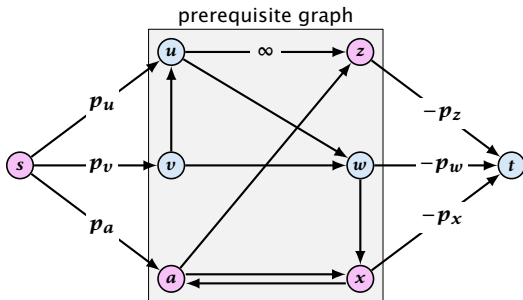
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**Proof.**

- ▶  $A$  is feasible because of capacity infinity edges.

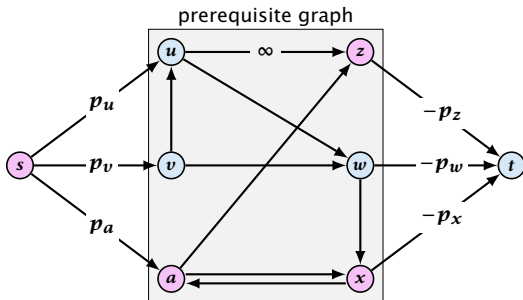


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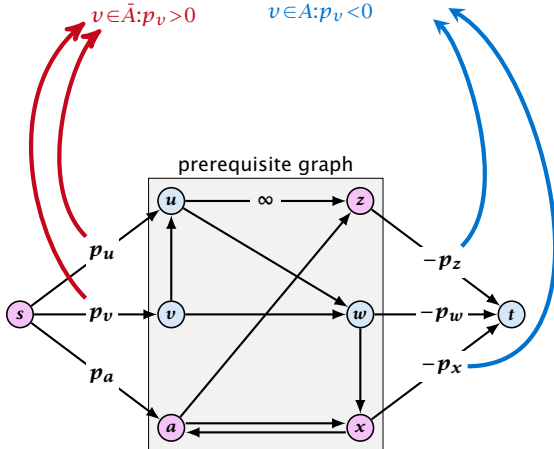
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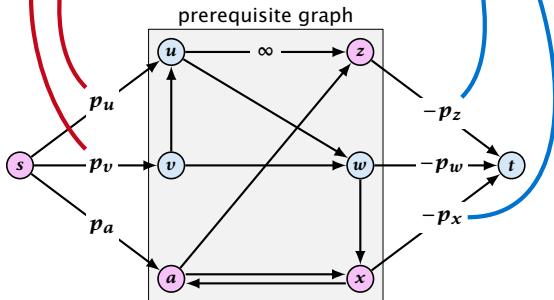
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$$= \sum_{v: p_v > 0} p_v - \sum_{v \in A} p_v$$





# Preflows

## Definition 26

An  $(s, t)$ -preflow is a function  $f : E \mapsto \mathbb{R}^+$  that satisfies

For each edge  $e \in E$

$$0 \leq f(e) \leq c(e)$$

For each vertex  $v \in V$

$$\sum_{e \in E^+} f(e) - \sum_{e \in E^-} f(e) \leq b(v)$$

For each vertex  $v \in V$

$$f(s) = f(t) = 0$$

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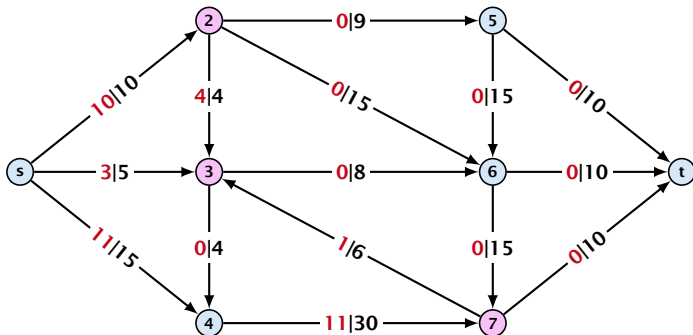
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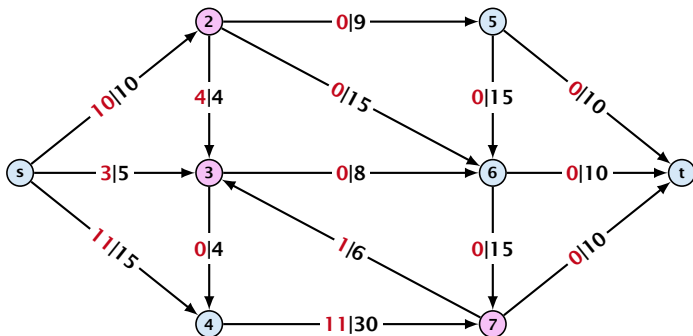
# Preflows

## Example 27



# Preflows

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A node that has  $\sum_{e \in \text{out}(v)} f(e) < \sum_{e \in \text{into}(v)} f(e)$  is called an **active node**.

# Preflows

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A **labelling** is a function  $\ell : V \rightarrow \mathbb{N}$ . It is **valid** for preflow  $f$  if

- ▶  $\ell(u) \leq \ell(v) + 1$  for all edges  $(u, v)$  in the residual graph  $G_f$  (only non-zero capacity edges!!!)

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- ▶  $\ell(u) \leq \ell(v) + 1$  for all edges  $(u, v)$  in the residual graph  $G_f$  (only non-zero capacity edges!!!)
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# Preflows

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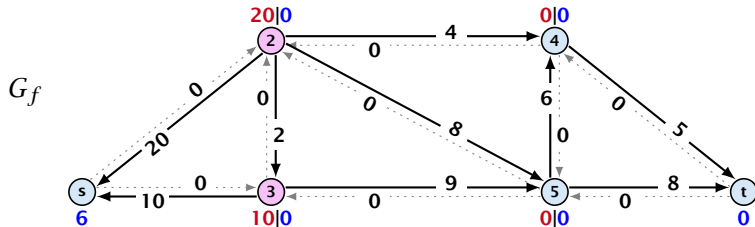
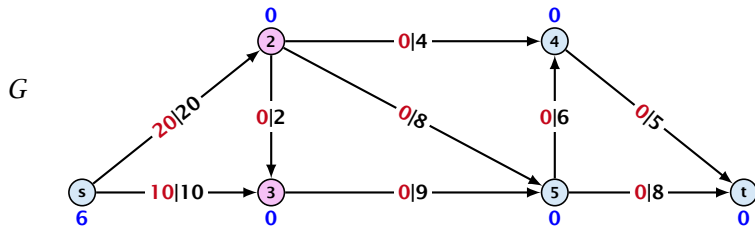
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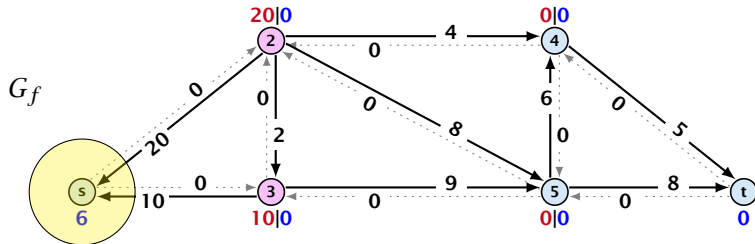
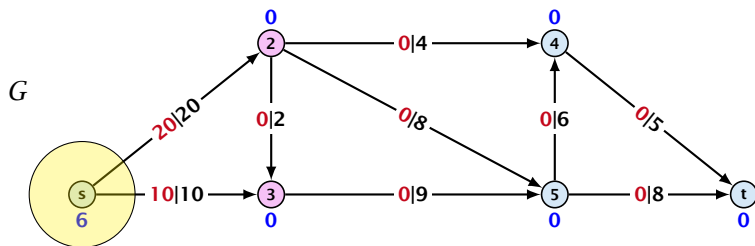
## Intuition:

The labelling can be viewed as a height function. Whenever the height from node  $u$  to node  $v$  decreases by more than 1 (i.e., it goes very steep downhill from  $u$  to  $v$ ), the corresponding edge must be saturated.

# Preflows



# Preflows



# Preflows

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## Lemma 28

A *preflow* that has a valid labelling saturates a cut.

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- ▶ There are  $n$  nodes but  $n + 1$  different labels from  $0, \dots, n$ .

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A *flow* that has a valid labelling is a maximum flow.

# Push Relabel Algorithms

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- ▶ start with some preflow and some valid labelling
- ▶ successively change the preflow while maintaining a valid labelling
- ▶ stop when you have a flow (i.e., no more active nodes)

## Changing a Preflow

An arc  $(u, v)$  with  $c_f(u, v) > 0$  in the residual graph is **admissible** if  $\ell(u) = \ell(v) + 1$  (i.e., it goes downwards w.r.t. labelling  $\ell$ ).

### The push operation

Consider an active node  $u$  with **excess flow**

$f(u) = \sum_{e \in \text{into}(u)} f(e) - \sum_{e \in \text{out}(u)} f(e)$  and suppose  $e = (u, v)$  is an admissible arc with residual capacity  $c_f(e)$ .

We can send flow  $\min\{c_f(e), f(u)\}$  along  $e$  and obtain a new preflow. The old labelling is still valid (!!!).

• The arc  $e$  is deleted from the residual graph

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the arc  $e$  is deleted from the residual graph
- ▶  **non-saturating push** :  $\min\{f(u), c_f(e)\} = f(u)$   
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Increasing the label of  $u$  by 1 results in a valid labelling.

- ▶ Edges  $(w, u)$  incoming to  $u$  still fulfill their constraint  $\ell(w) \leq \ell(u) + 1$ .
- ▶ An outgoing edge  $(u, w)$  had  $\ell(u) < \ell(w) + 1$  before since it was not admissible. Now:  $\ell(u) \leq \ell(w) + 1$ .

# Push Relabel Algorithms

## Intuition:

We want to send flow downwards, since the source has a height/label of  $n$  and the target a height/label of  $0$ . If we see an active node  $u$  with an admissible arc we push the flow at  $u$  towards the other end-point that has a lower height/label. If we do not have an admissible arc but excess flow into  $u$  it should roughly mean that the level/height/label of  $u$  should rise. (If we consider the flow to be water than this would be natural).

Note that the above intuition is very incorrect as the labels are integral, i.e., they cannot really be seen as the height of a node.

# Reminder

- ▶ In a **preflow** nodes may not fulfill conservation constraints; a node may have more incoming flow than outgoing flow.
- ▶ Such a node is called **active**.
- ▶ A labelling is **valid** if for every edge  $(u, v)$  in the residual graph  $\ell(u) \leq \ell(v) + 1$ .
- ▶ An arc  $(u, v)$  in residual graph is **admissible** if  $\ell(u) = \ell(v) + 1$ .
- ▶ A **saturating push** along  $e$  pushes an amount of  $c(e)$  flow along the edge, thereby saturating the edge (and making it disappear from the residual graph).
- ▶ A **non-saturating push** along  $e = (u, v)$  pushes a flow of  $f(u)$ , where  $f(u)$  is the **excess flow** of  $u$ . This makes  $u$  inactive.

# Push Relabel Algorithms

## Algorithm 3 $\text{maxflow}(G, s, t, c)$

```
1: find initial preflow  $f$ 
2: while there is active node  $u$  do
3:     if there is admiss. arc  $e$  out of  $u$  then
4:          $\text{push}(G, e, f, c)$ 
5:     else
6:          $\text{relabel}(u)$ 
7: return  $f$ 
```

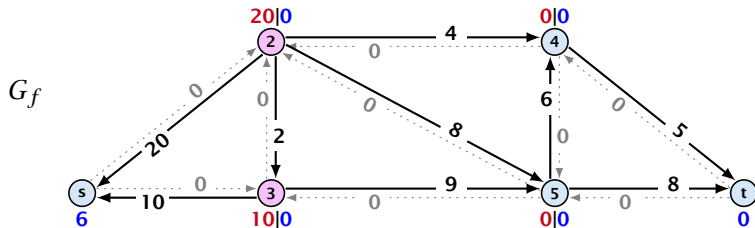
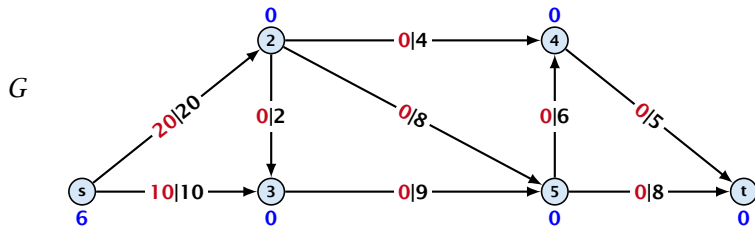
# Push Relabel Algorithms

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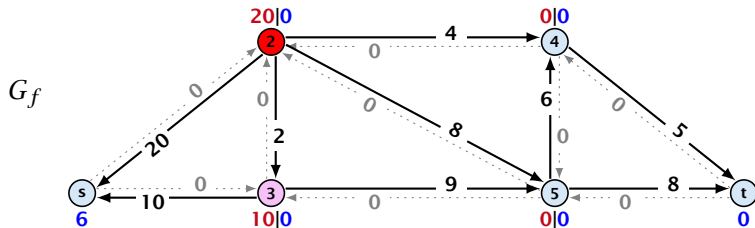
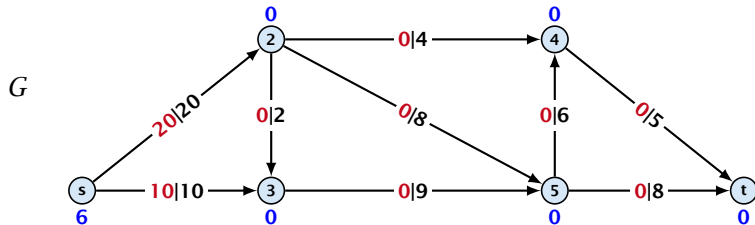
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In the following example we always stick to the same active node  $u$  until it becomes inactive but this is not required.

# Preflow Push Algorithm



# Preflow Push Algorithm

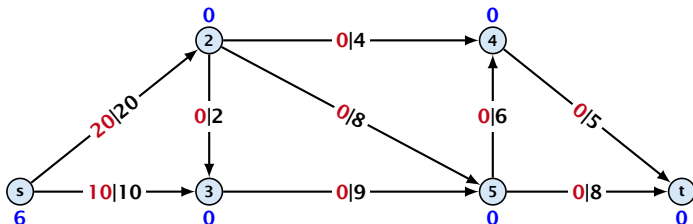




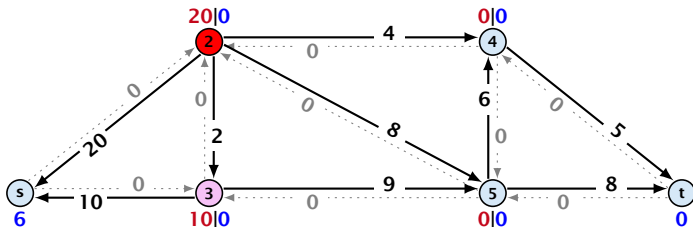
# Preflow Push Algorithm

relabel

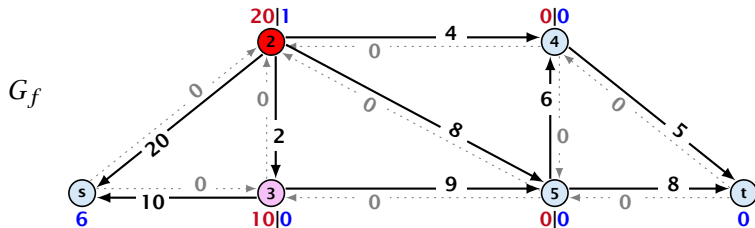
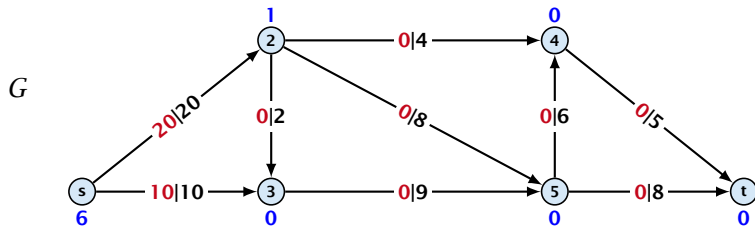
$G$



$G_f$

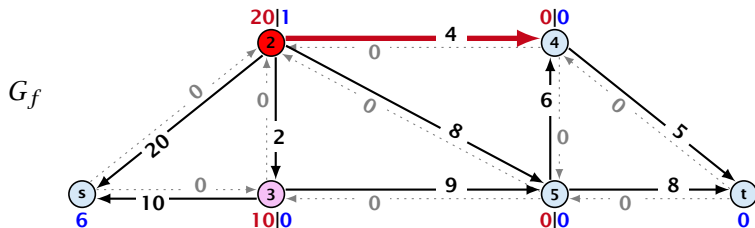
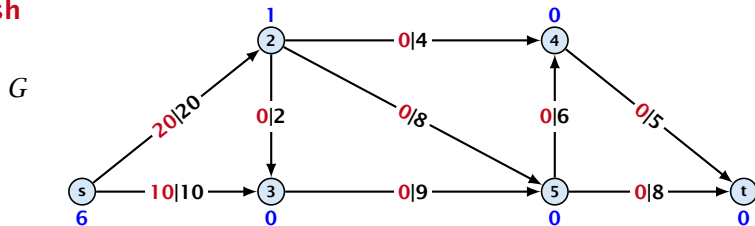


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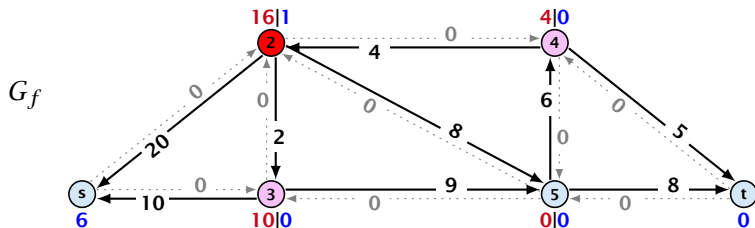
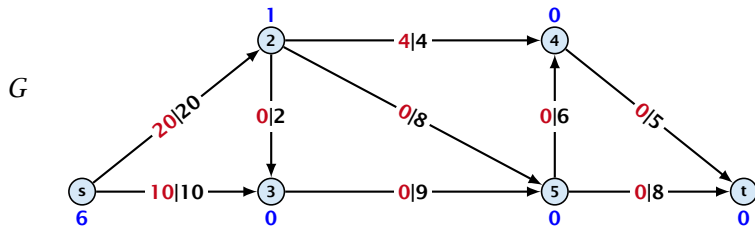


# Preflow Push Algorithm

push



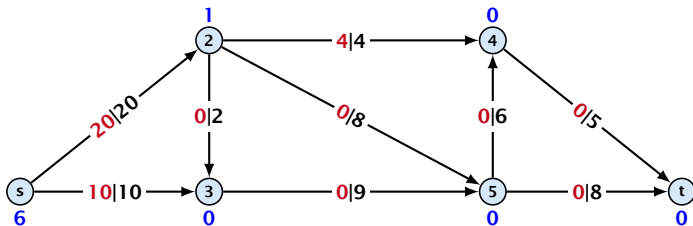
# Preflow Push Algorithm



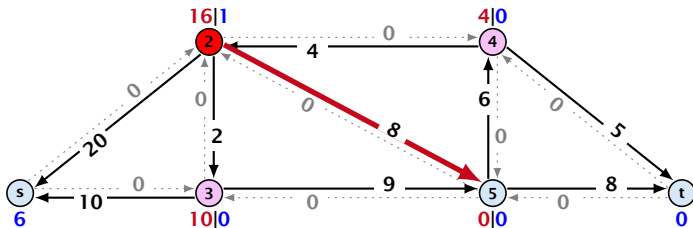
# Preflow Push Algorithm

push

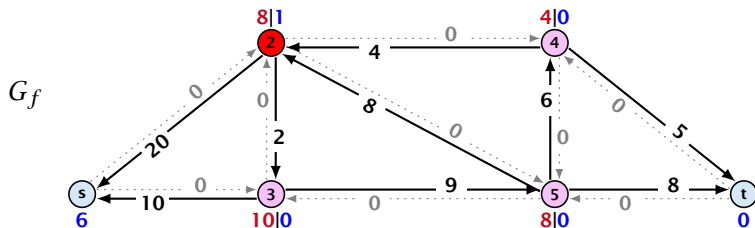
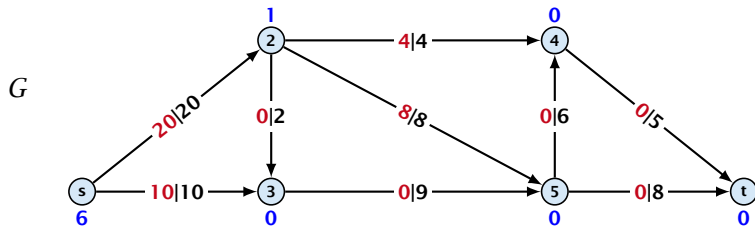
$G$



$G_f$



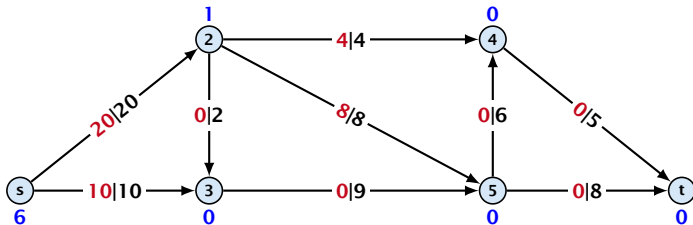
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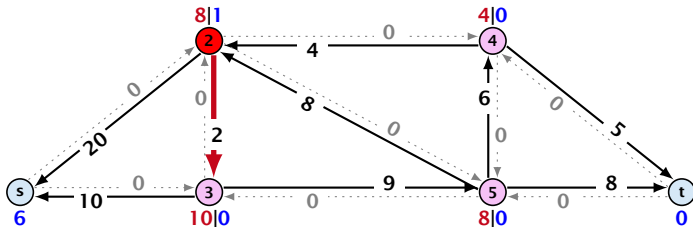
# Preflow Push Algorithm

push

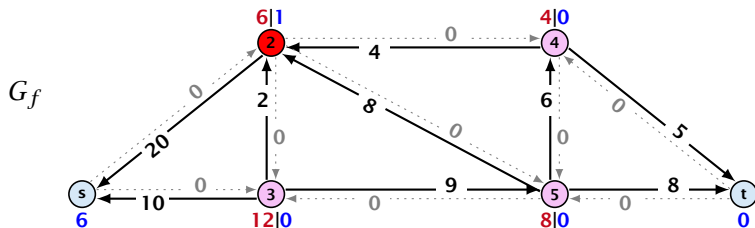
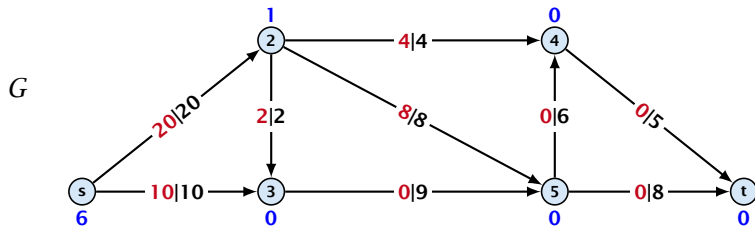
$G$



$G_f$



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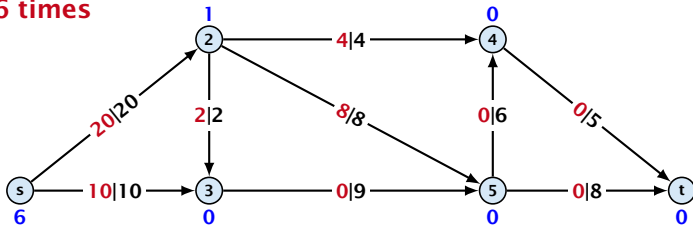




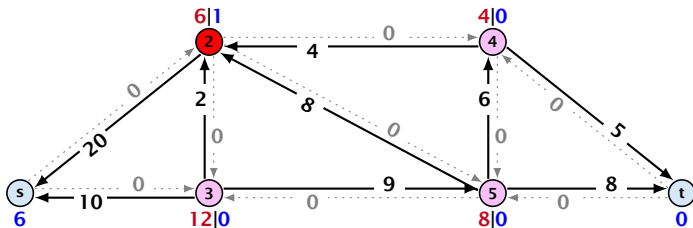
# Preflow Push Algorithm

relabel 6 times

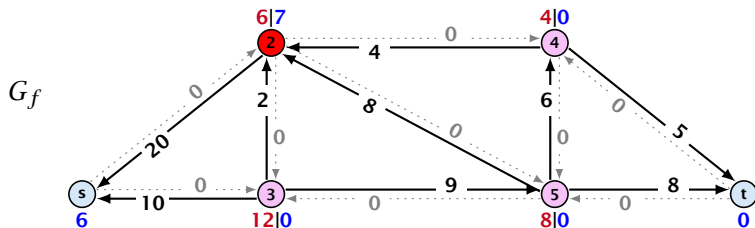
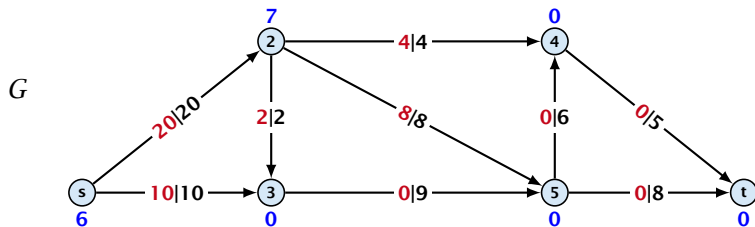
$G$



$G_f$



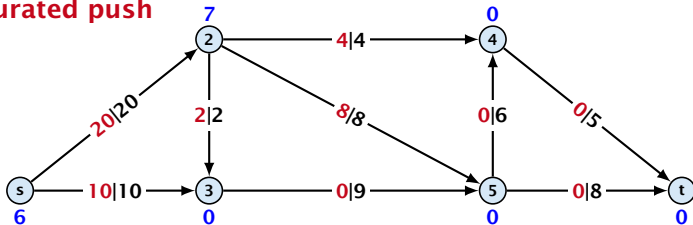
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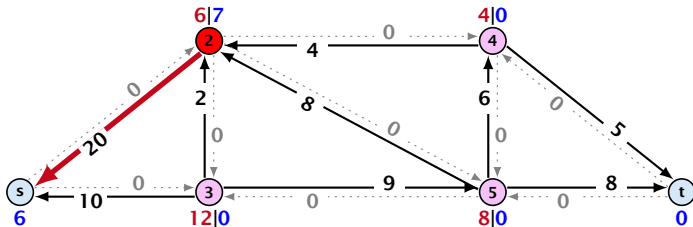
# Preflow Push Algorithm

non-saturated push

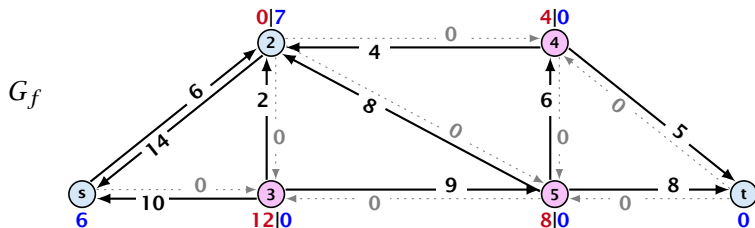
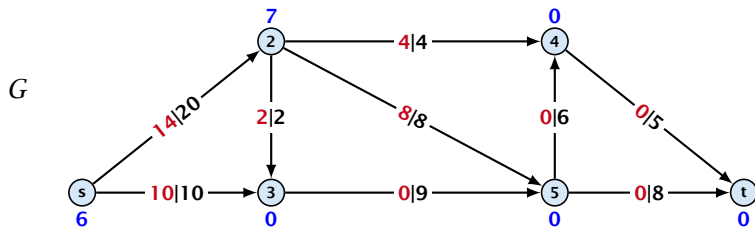
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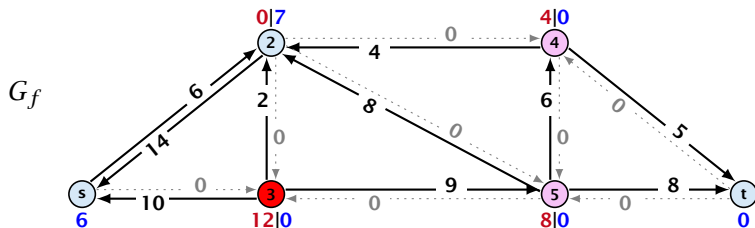
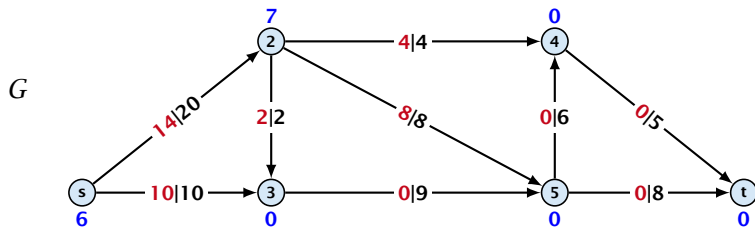
$G_f$



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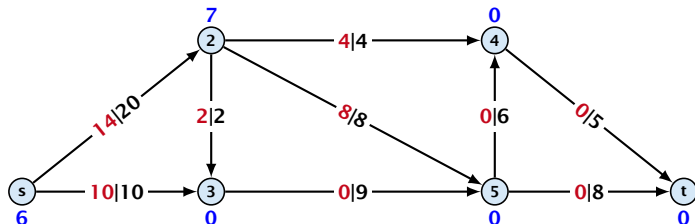
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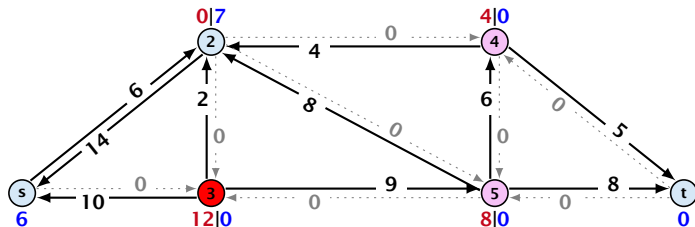
# Preflow Push Algorithm

relabel

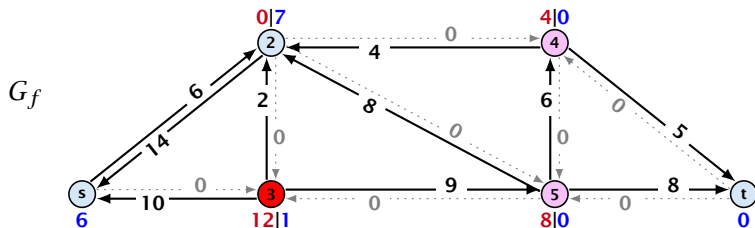
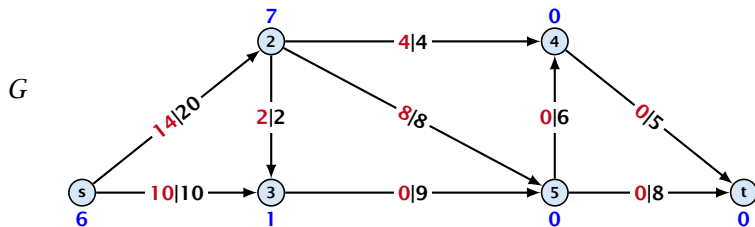
$G$



$G_f$



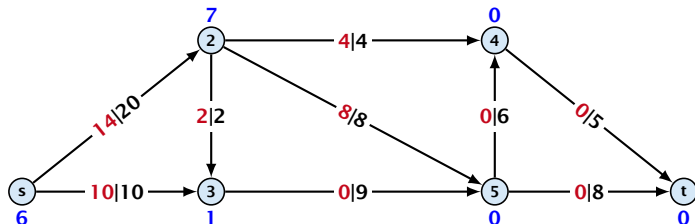
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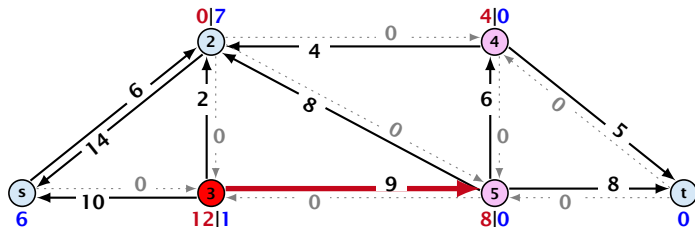
# Preflow Push Algorithm

push

$G$

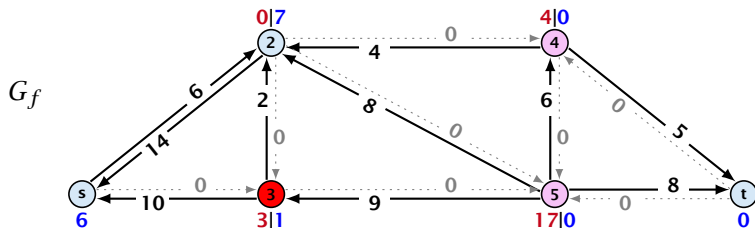
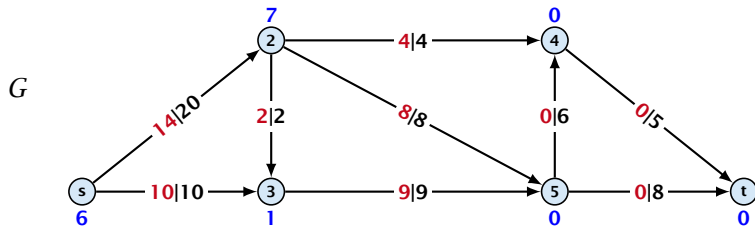


$G_f$





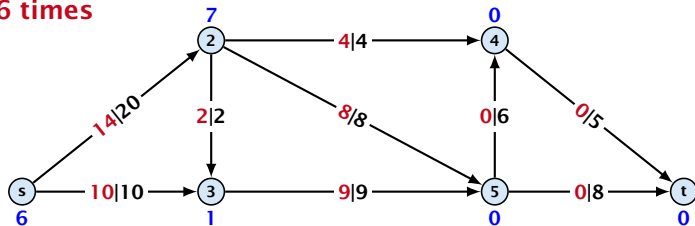
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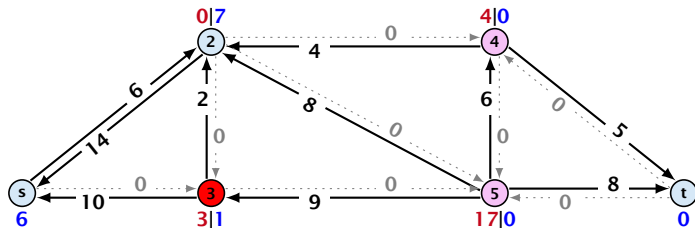
# Preflow Push Algorithm

relabel 6 times

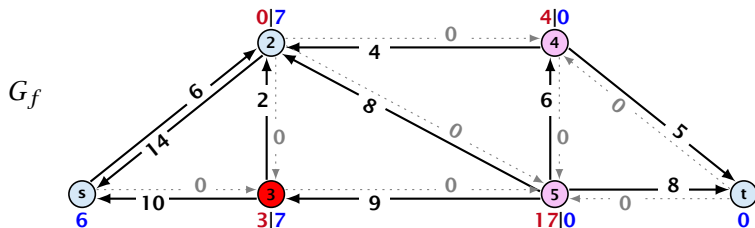
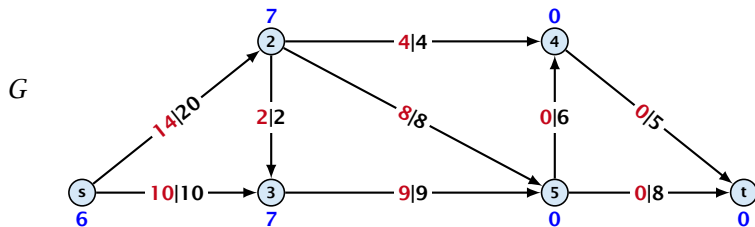
$G$



$G_f$



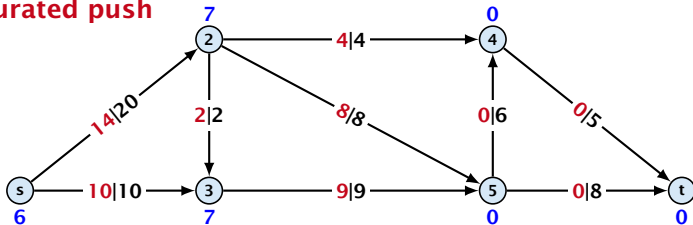
# Preflow Push Algorithm



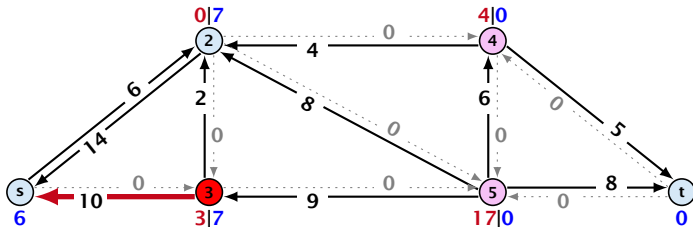
# Preflow Push Algorithm

non-saturated push

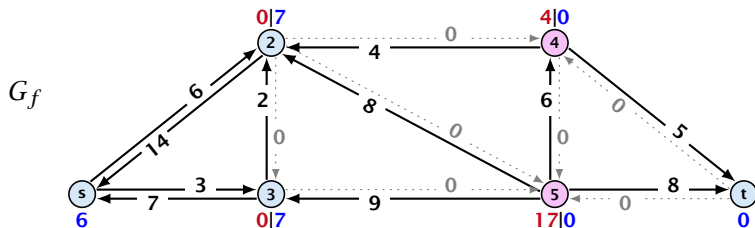
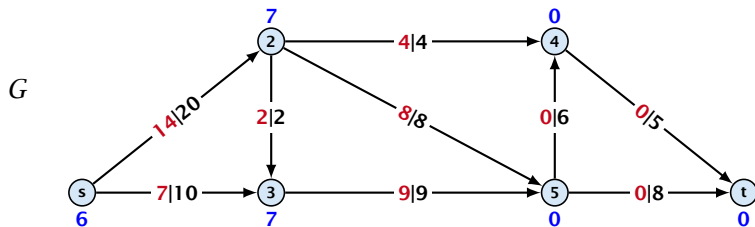
$G$



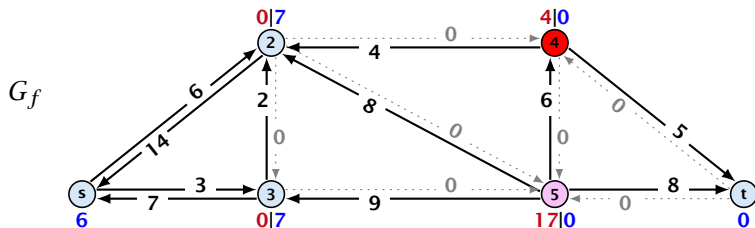
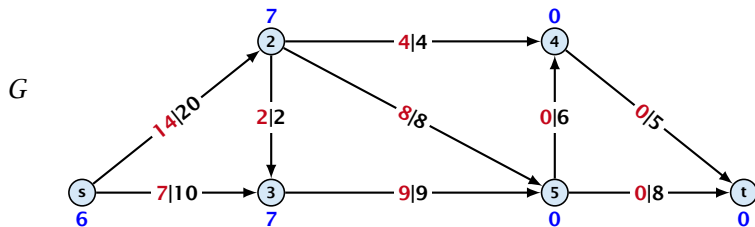
$G_f$



# Preflow Push Algorithm



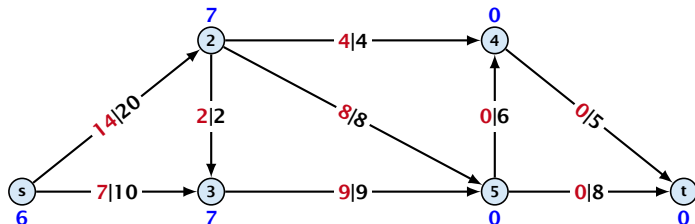
# Preflow Push Algorithm



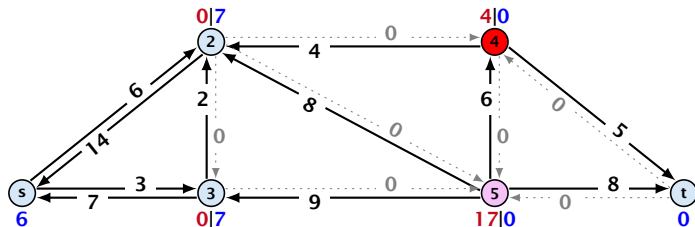
# Preflow Push Algorithm

relabel

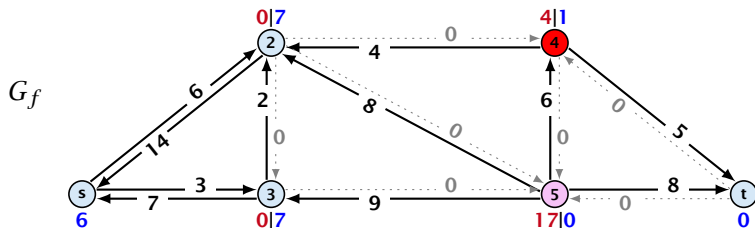
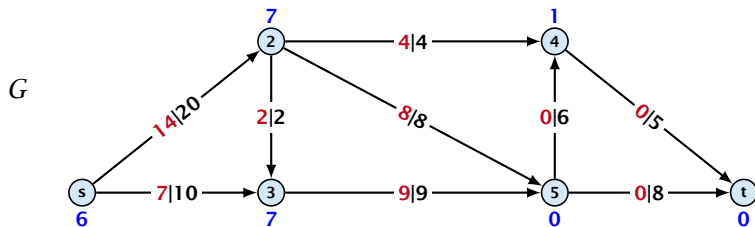
$G$



$G_f$



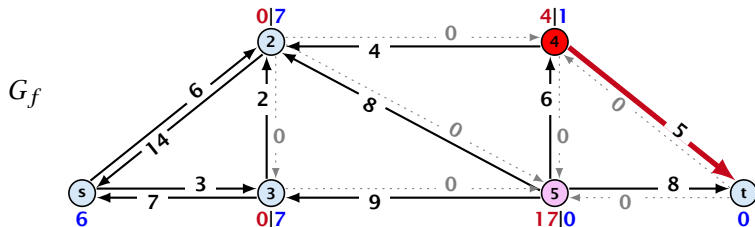
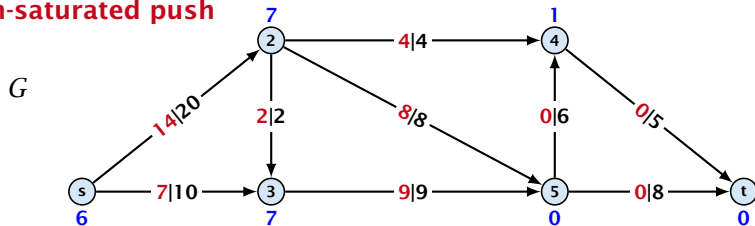
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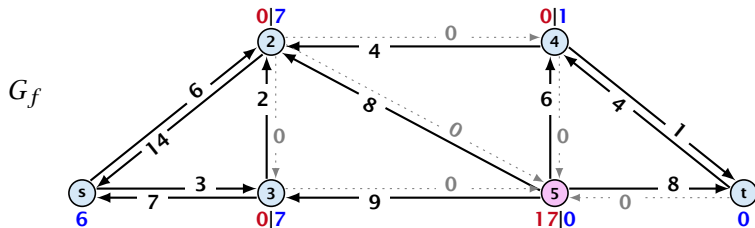
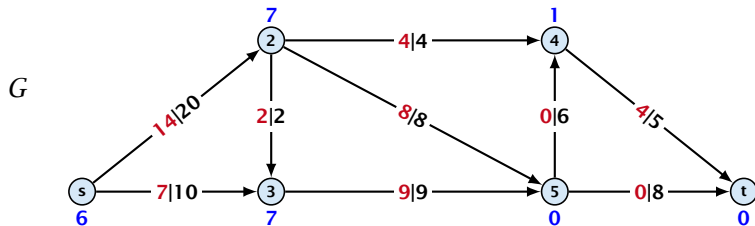


# Preflow Push Algorithm

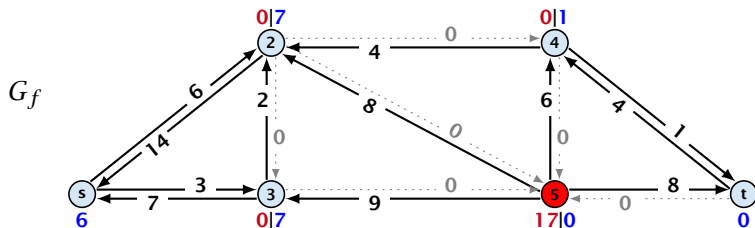
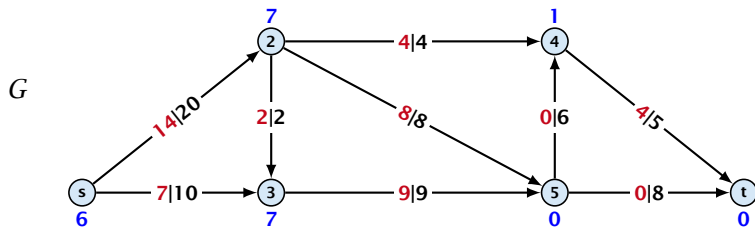
non-saturated push



# Preflow Push Algorithm



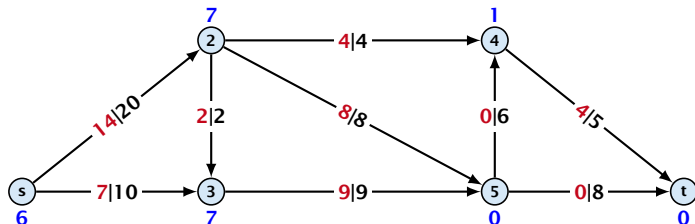
# Preflow Push Algorithm



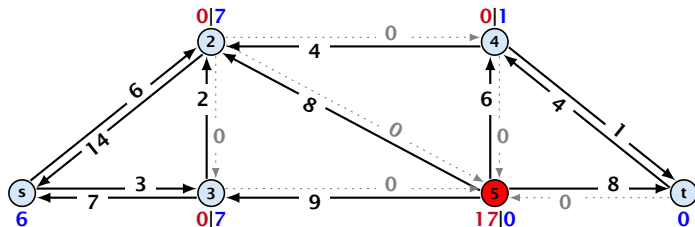
# Preflow Push Algorithm

relabel

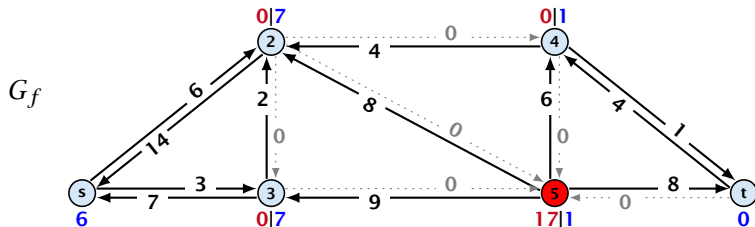
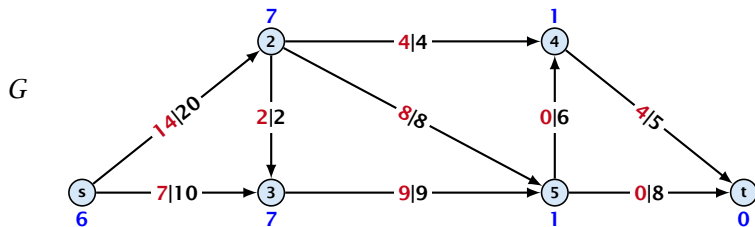
$G$



$G_f$



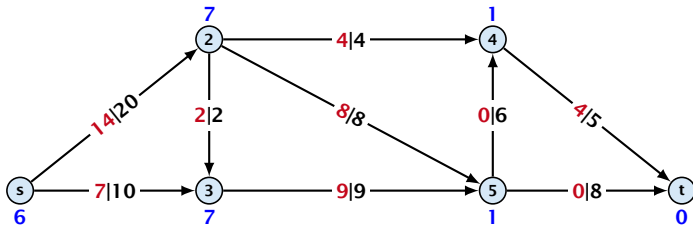
# Preflow Push Algorithm



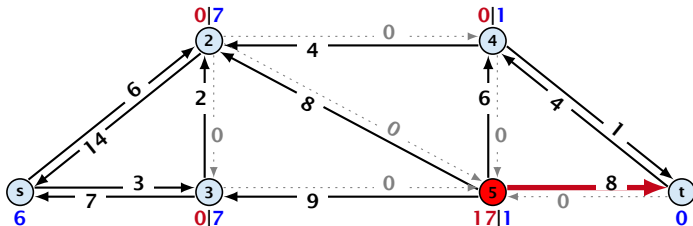
# Preflow Push Algorithm

push

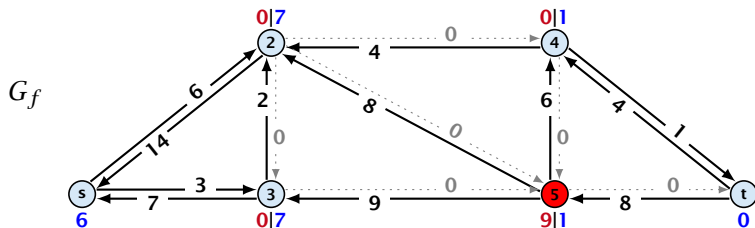
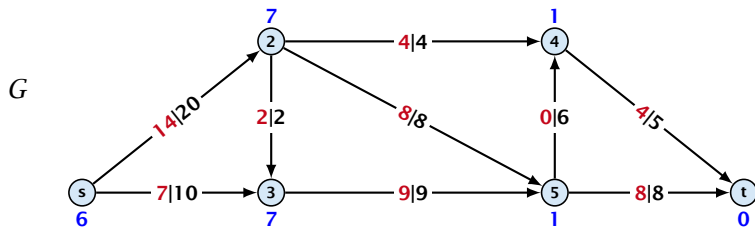
$G$



$G_f$



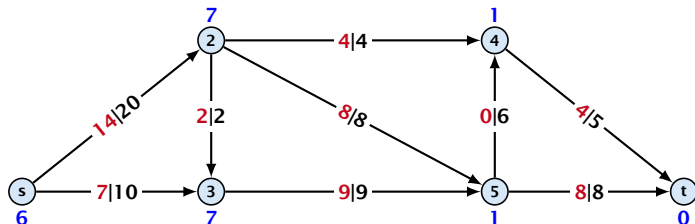
# Preflow Push Algorithm



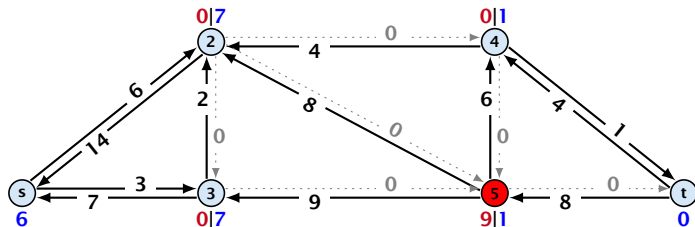
# Preflow Push Algorithm

relabel

$G$

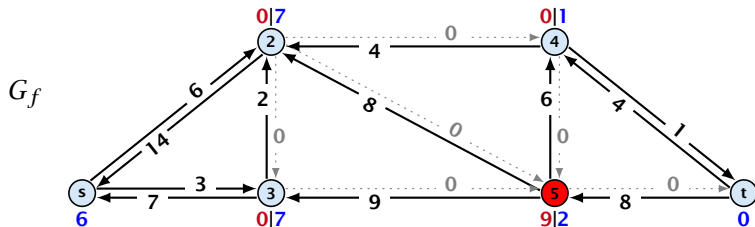
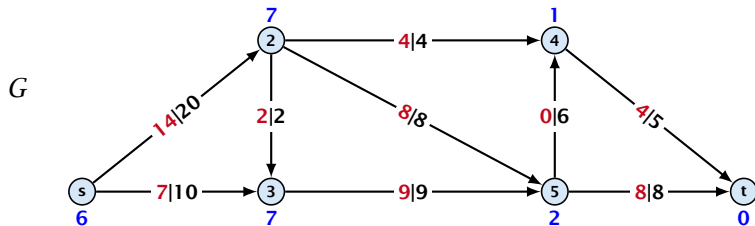


$G_f$





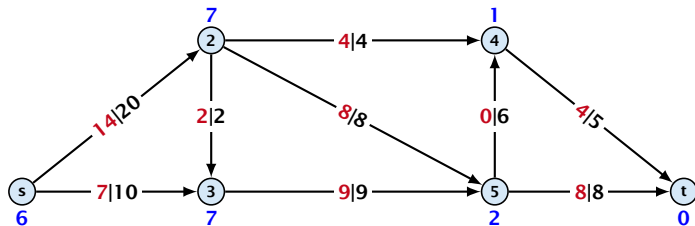
# Preflow Push Algorithm



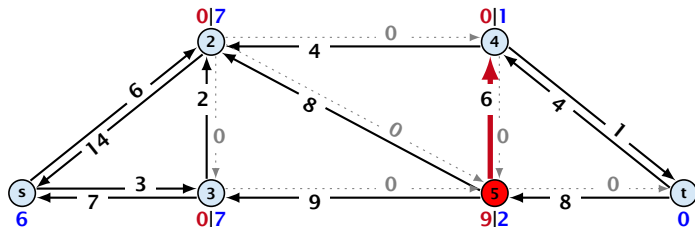
# Preflow Push Algorithm

push

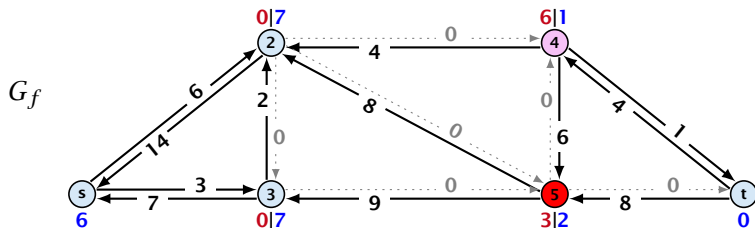
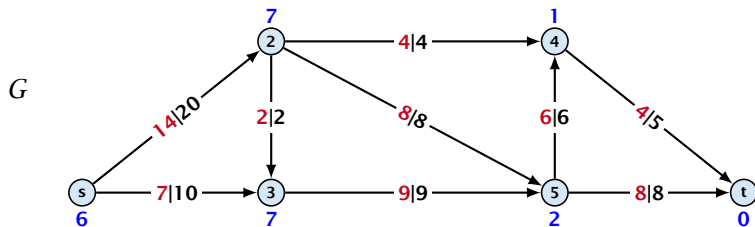
$G$



$G_f$



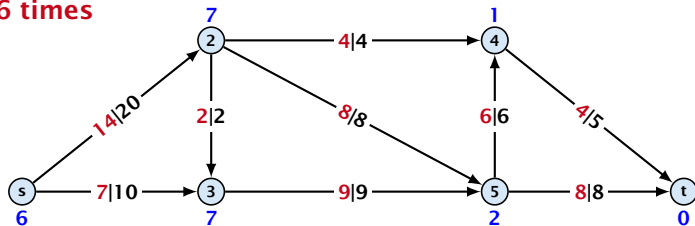
# Preflow Push Algorithm



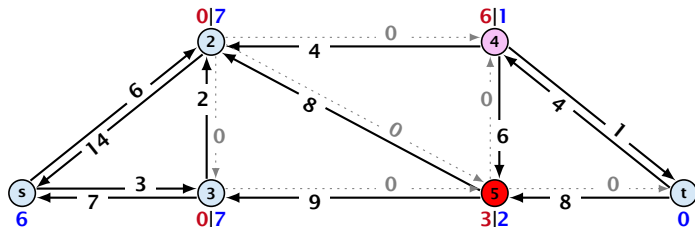
# Preflow Push Algorithm

relabel 6 times

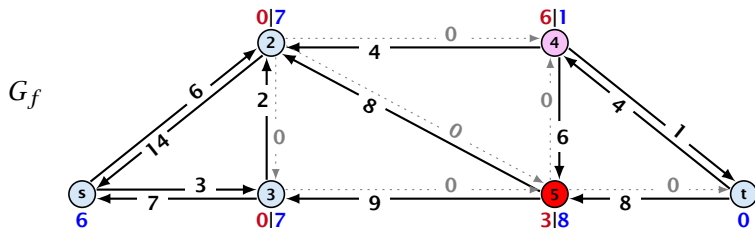
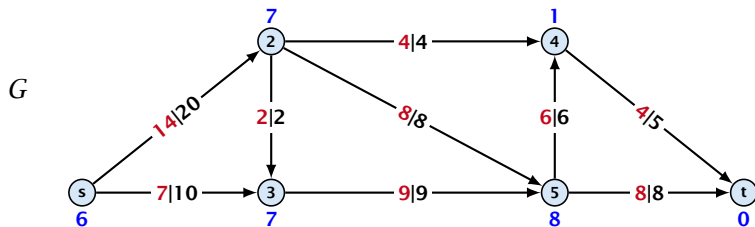
$G$



$G_f$



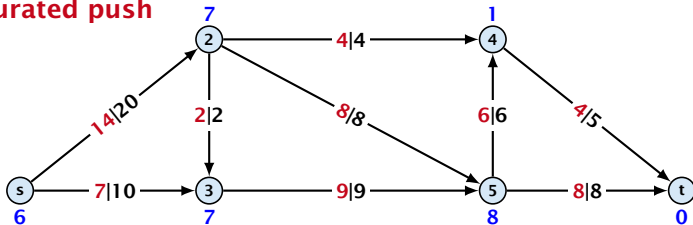
# Preflow Push Algorithm



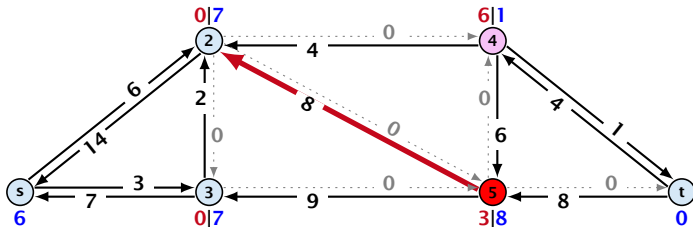
# Preflow Push Algorithm

non-saturated push

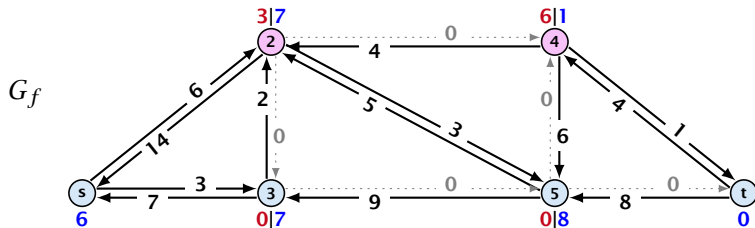
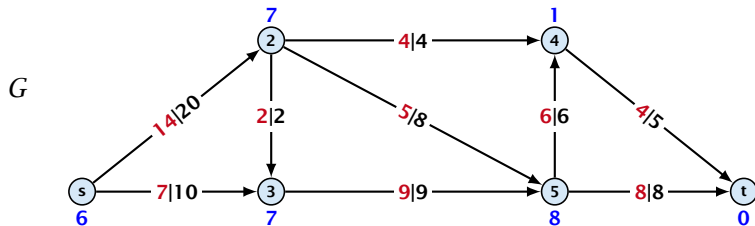
$G$



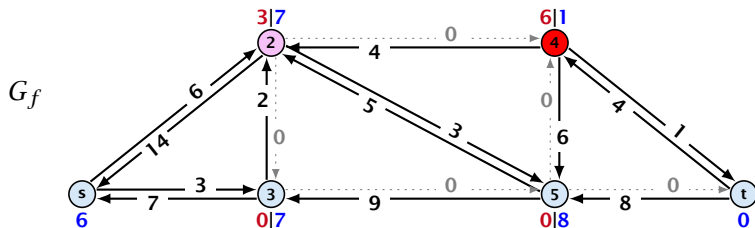
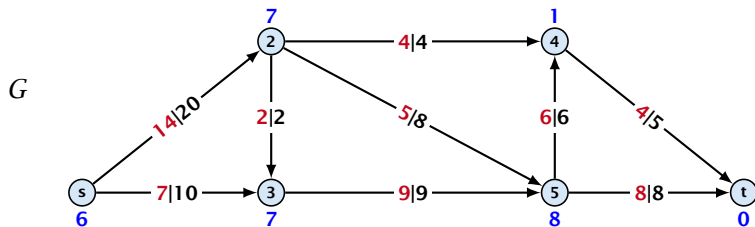
$G_f$



# Preflow Push Algorithm



# Preflow Push Algorithm

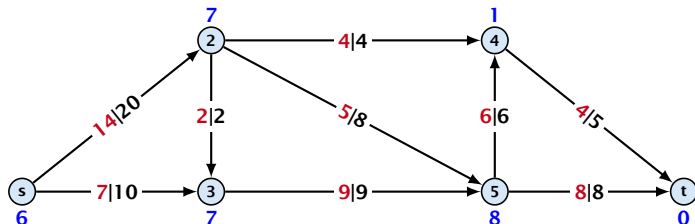




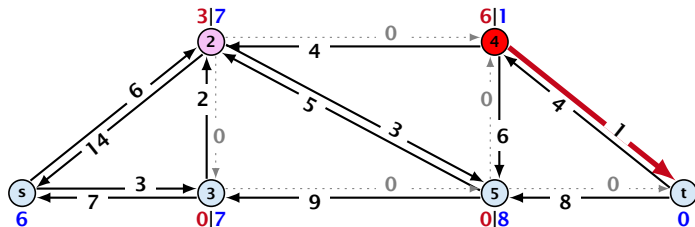
# Preflow Push Algorithm

push

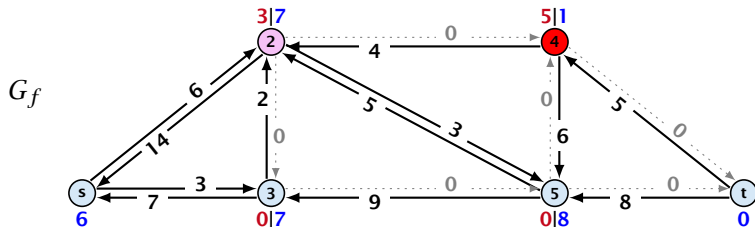
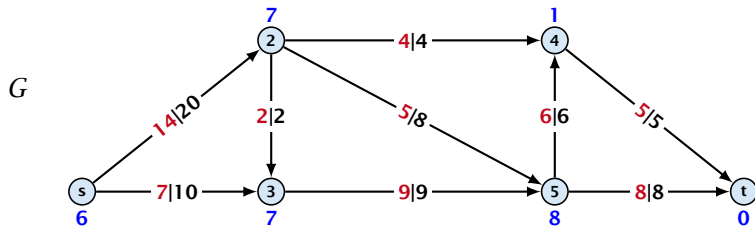
$G$



$G_f$



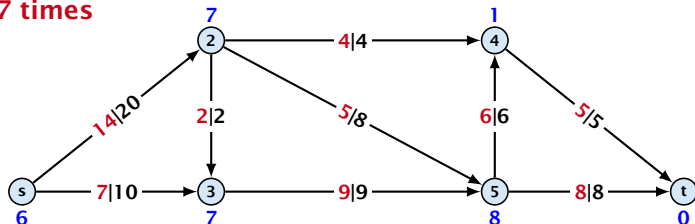
# Preflow Push Algorithm



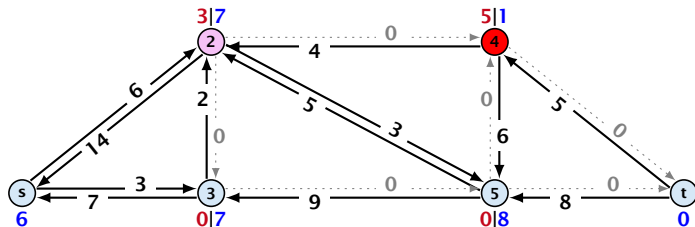
# Preflow Push Algorithm

relabel 7 times

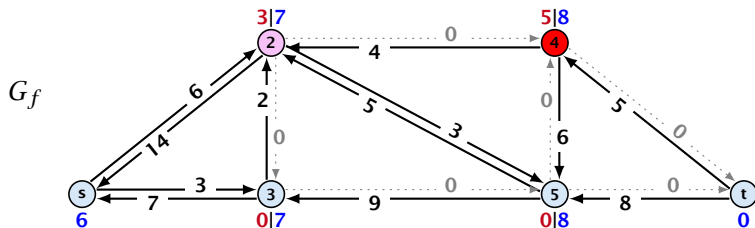
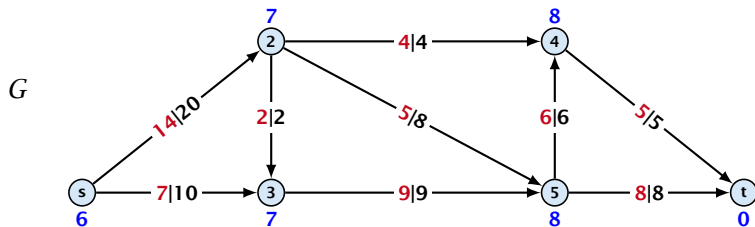
$G$



$G_f$

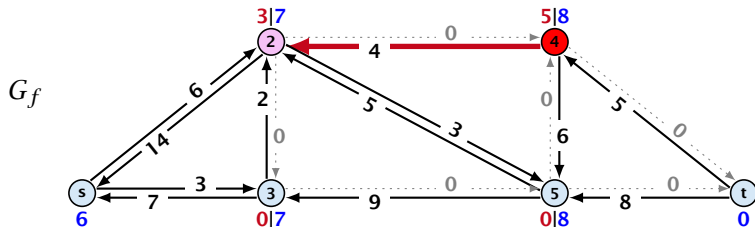
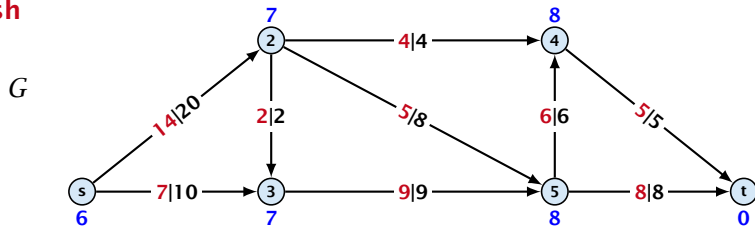


# Preflow Push Algorithm

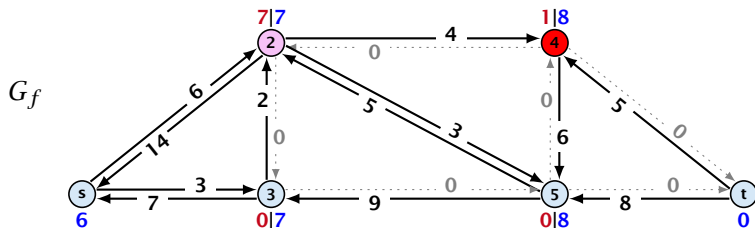
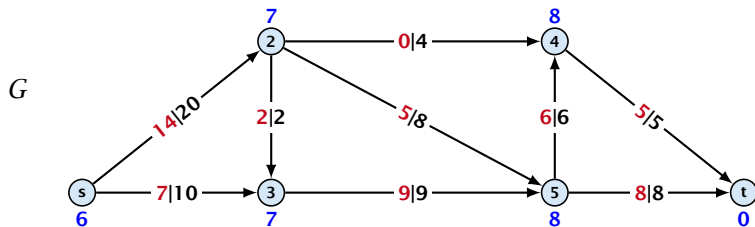


# Preflow Push Algorithm

push



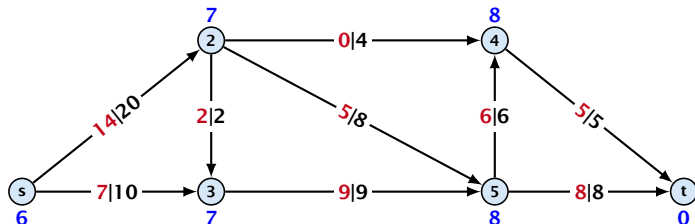
# Preflow Push Algorithm



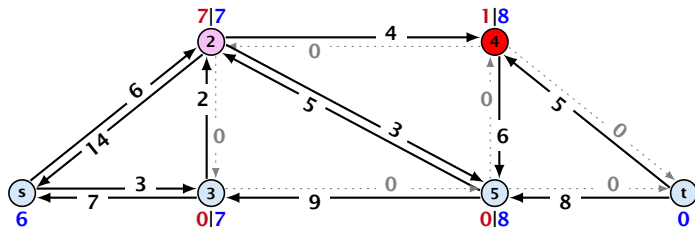
# Preflow Push Algorithm

relabel

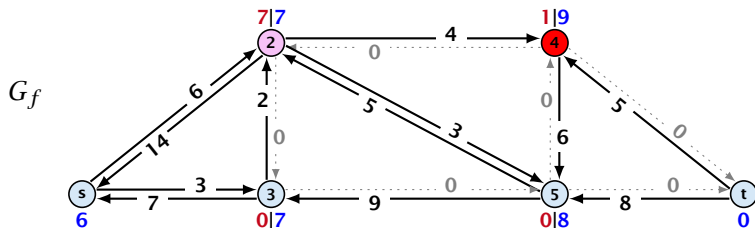
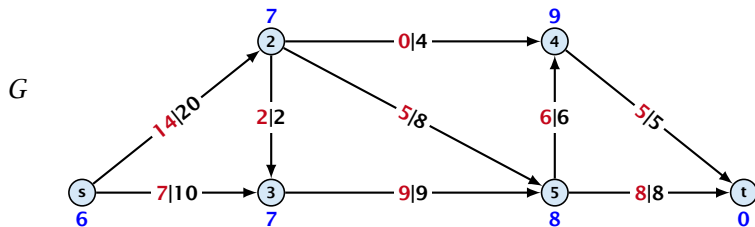
$G$



$G_f$



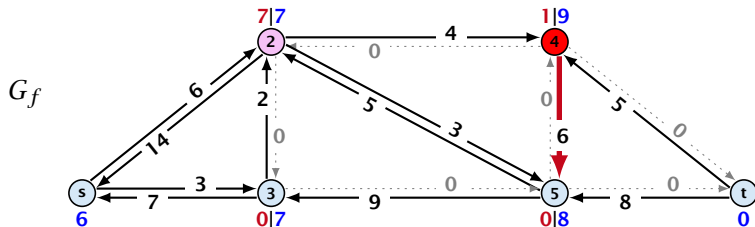
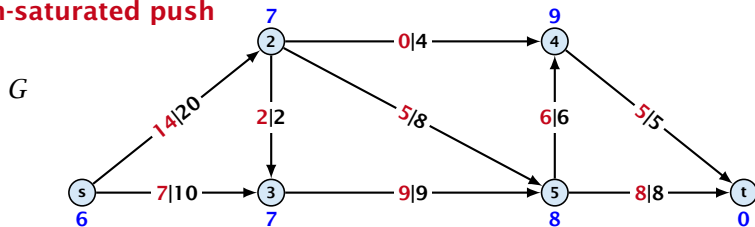
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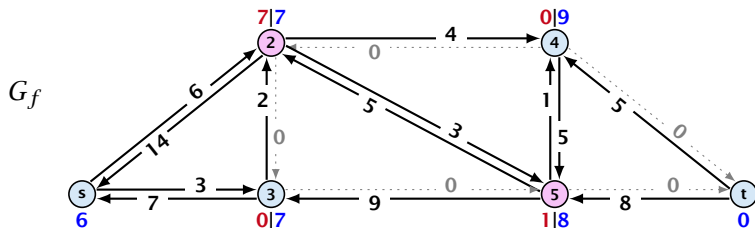
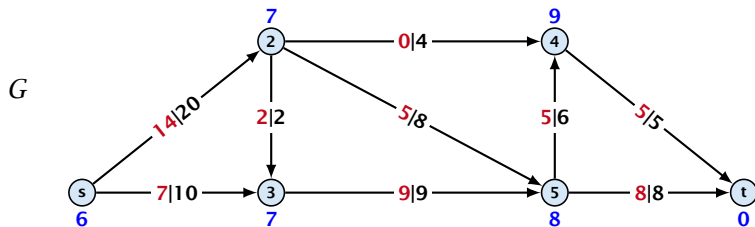


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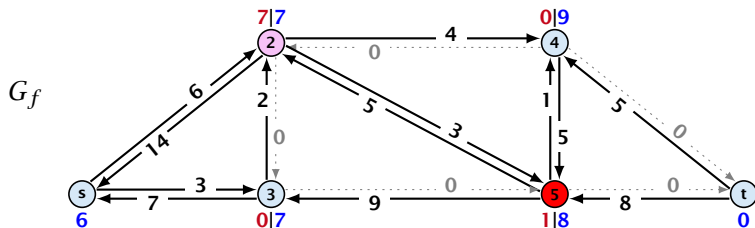
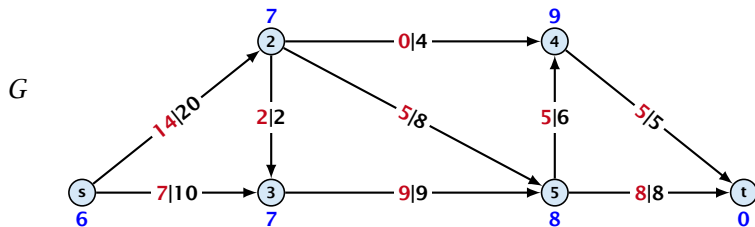
## non-saturated push



# Preflow Push Algorithm



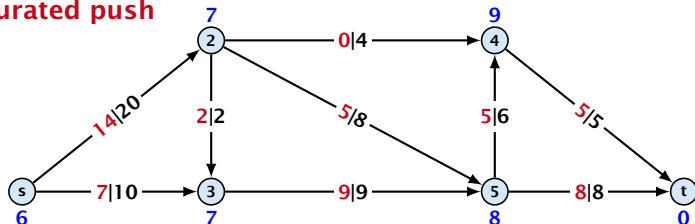
# Preflow Push Algorithm



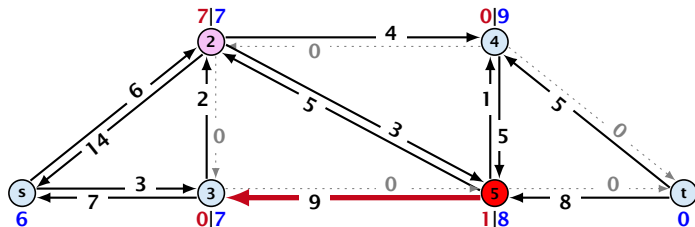
# Preflow Push Algorithm

non-saturated push

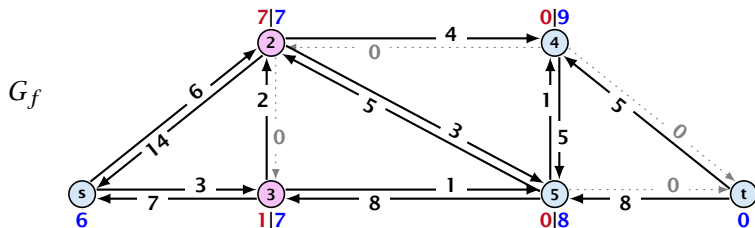
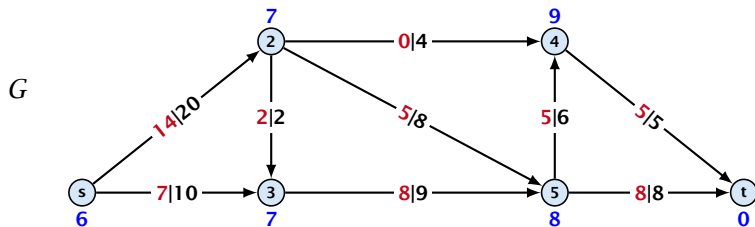
$G$



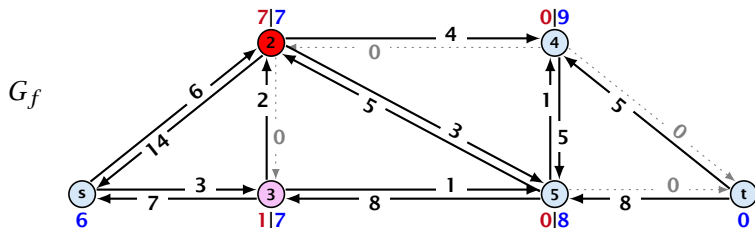
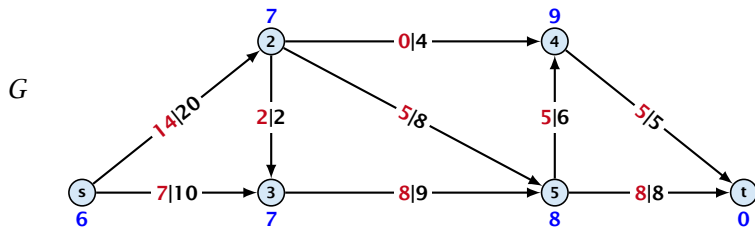
$G_f$



# Preflow Push Algorithm

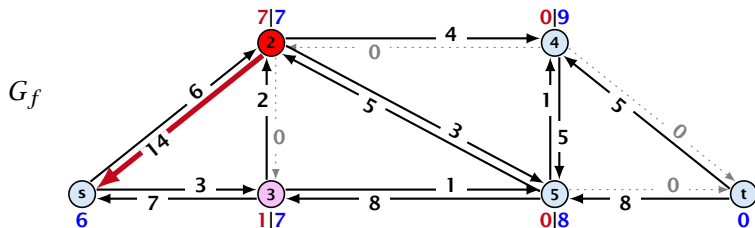
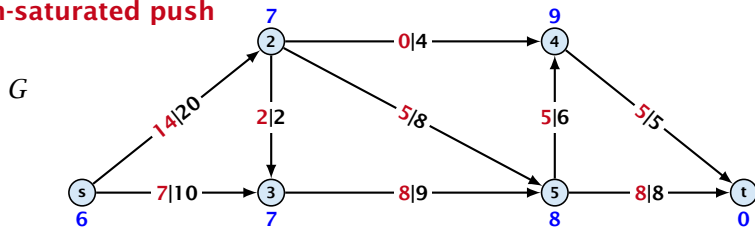


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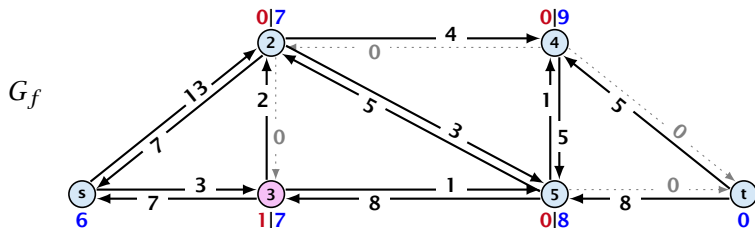
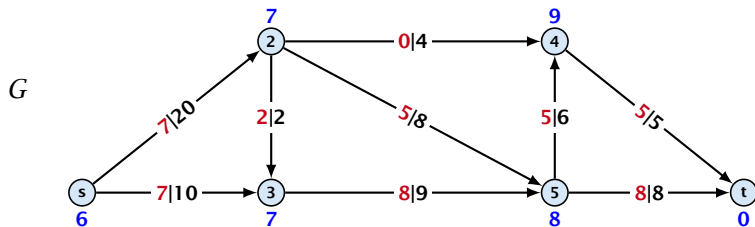


# Preflow Push Algorithm

non-saturated push

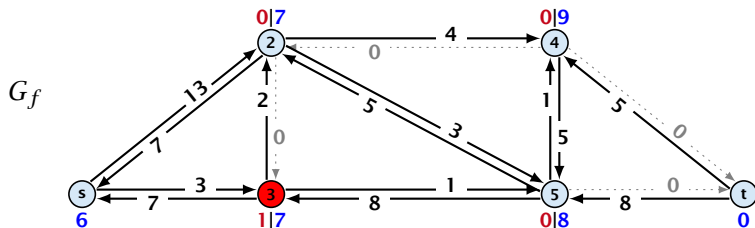
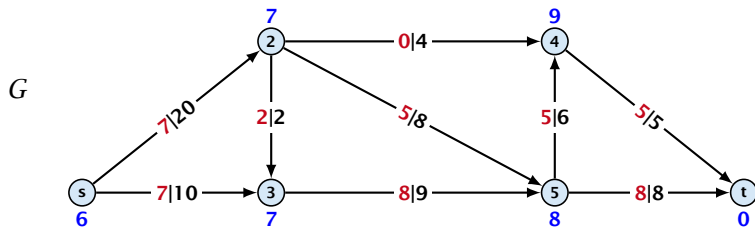


# Preflow Push Algorithm





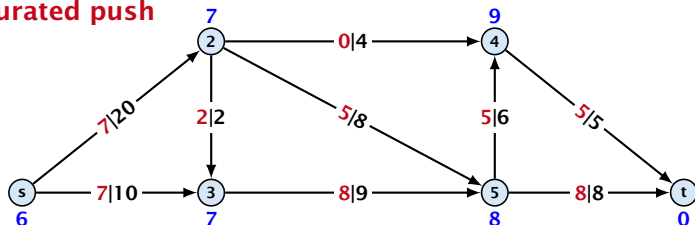
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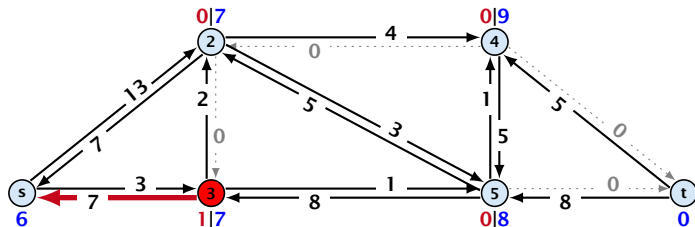
# Preflow Push Algorithm

## non-saturated push

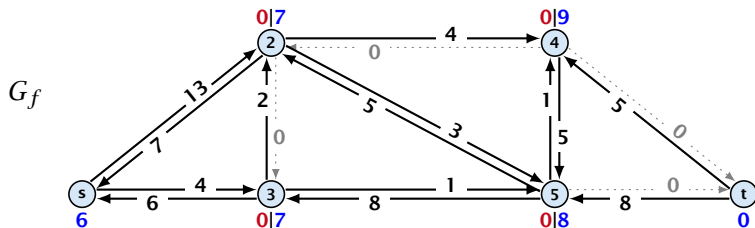
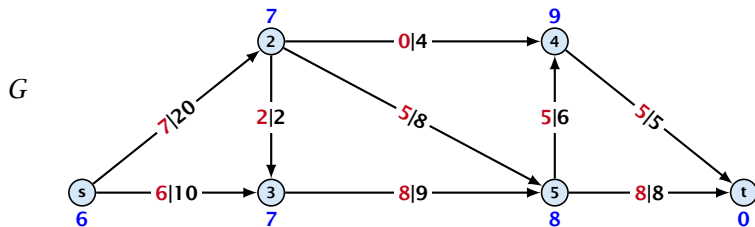
$G$



$G_f$



# Preflow Push Algorithm



# Analysis

## Lemma 30

*An active node has a path to  $s$  in the residual graph.*

# Analysis

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### Proof.

- ▶ Let  $A$  denote the set of nodes that can reach  $s$ , and let  $B$  denote the remaining nodes. Note that  $s \in A$ .

# Analysis

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- ▶ Let  $f(B) = \sum_{v \in B} f(v)$  be the excess flow of all nodes in  $B$ .



Let  $f : E \rightarrow \mathbb{R}_0^+$  be a preflow. We introduce the notation

$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

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We have

$$f(B)$$

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$$f(B) = \sum_{b \in B} f(b)$$

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$$f(x, y) = \begin{cases} 0 & (x, y) \notin E \\ f((x, y)) & (x, y) \in E \end{cases}$$

We have

$$\begin{aligned} f(B) &= \sum_{b \in B} f(b) \\ &= \sum_{b \in B} \left( \sum_{v \in V} f(v, b) - \sum_{v \in V} f(b, v) \right) \end{aligned}$$

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Hence, the excess flow  $f(b)$  must be 0 for every node  $b \in B$ .

# Analysis

## Lemma 31

*The label of a node cannot become larger than  $2n - 1$ .*

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### Proof.

- ▶ When increasing the label at a node  $u$  there exists a path from  $u$  to  $s$  of length at most  $n - 1$ . Along each edge of the path the height/label can at most drop by 1, and the label of the source is  $n$ .



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## Lemma 32

*There are only  $\mathcal{O}(n^2)$  relabel operations.*

# Analysis

## Lemma 33

The number of *saturating pushes* performed is at most  $\mathcal{O}(mn)$ .

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- ▶ Hence, the edge  $(u, v)$  is deleted from the residual graph.

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The number of *saturating pushes* performed is at most  $\mathcal{O}(mn)$ .

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- ▶ Suppose that we just made a saturating push along  $(u, v)$ .
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- ▶ Currently,  $\ell(u) = \ell(v) + 1$ , as we only make pushes along admissible edges.

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- ▶ For a push from  $v$  to  $u$  the edge  $(v, u)$  must become admissible. The label of  $v$  must increase by at least 2.
- ▶ Since the label of  $v$  is at most  $2n - 1$ , there are at most  $n$  pushes along  $(u, v)$ .



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The number of *non-saturating pushes* performed is at most  $\mathcal{O}(n^2m)$ .

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- ▶ Define a potential function  $\Phi(f) = \sum_{\text{active nodes } v} \ell(v)$
- ▶ A saturating push increases  $\Phi$  by  $\leq 2n$  (when the target node becomes active it may contribute at most  $2n$  to the sum).

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- ▶ A relabel increases  $\Phi$  by at most 1.

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- ▶ A non-saturating push decreases  $\Phi$  by at least  $1$  as the node that is pushed from becomes inactive and has a label that is strictly larger than the target.
- ▶ Hence,

$$\begin{aligned} \# \text{non-saturating\_pushes} &\leq \# \text{relabels} + 2n \cdot \# \text{saturating\_pushes} \\ &\leq \mathcal{O}(n^2m) . \end{aligned}$$

## Theorem 35

*There is an implementation of the generic push relabel algorithm with running time  $\mathcal{O}(n^2m)$ .*

# Analysis

## Proof:

For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

A push along an edge  $(u, v)$  can be performed in constant time

- check whether edge  $(u, v)$  needs to be added to  $E(u)$
- check whether  $(u, v)$  needs to be deleted (aftering push)
- check whether  $v$  becomes inactive and has to be deleted from the set of active nodes

A relabel at a node  $u$  can be performed in time  $\mathcal{O}(n)$

- check for all outgoing edges if they become inadmissible
- check for all incoming edges if they become inadmissible



# Analysis

## Proof:

For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

A push along an edge  $(u, v)$  can be performed in constant time  
check whether  $v$  is active (if not, add  $v$  to the set of active nodes)  
check whether  $u$  needs to be relabel (returning push)  
check whether  $u$  becomes inactive and has to be deleted  
from the set of active nodes

A relabel at a node  $u$  can be performed in time  $\mathcal{O}(n)$   
check for all outgoing edges if they become admissible  
check for all incoming edges if they become inadmissible

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## Proof:

For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

A push along an edge  $(u, v)$  can be performed in constant time

- ▶ check whether edge  $(v, u)$  needs to be added to  $G_f$
- ▶ check whether  $(u, v)$  needs to be deleted (saturating push)
- ▶ check whether  $u$  becomes inactive and has to be deleted from the set of active nodes

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A relabel at a node  $u$  can be performed in time  $\mathcal{O}(n)$

- ▶ check for all outgoing edges if they become admissible
- ▶ check for all incoming edges if they become non-admissible

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For every node maintain a list of admissible edges starting at that node. Further maintain a list of active nodes.

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- ▶ check whether  $(u, v)$  needs to be deleted (saturating push)
- ▶ check whether  $u$  becomes inactive and has to be deleted from the set of active nodes

A relabel at a node  $u$  can be performed in time  $\mathcal{O}(n)$

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## Analysis

For special variants of push relabel algorithms we organize the neighbours of a node into a linked list (possible neighbours in the residual graph  $G_f$ ). Then we use the discharge-operation:

### Algorithm 4 discharge( $u$ )

```
1: while  $u$  is active do  
2:    $v \leftarrow u.current\text{-neighbour}$   
3:   if  $v = \text{null}$  then  
4:     relabel( $u$ )  
5:      $u.current\text{-neighbour} \leftarrow u.neighbour\text{-list-head}$   
6:   else  
7:     if  $(u, v)$  admissable then push( $u, v$ )  
8:     else  $u.current\text{-neighbour} \leftarrow v.next\text{-in-list}$ 
```

Note that  $u.current\text{-neighbour}$  is a global variable. It is only changed within the discharge routine, but keeps its value between consecutive calls to discharge.

## Lemma 36

If  $v = \text{null}$  in Line 3, then there is no outgoing admissible edge from  $u$ .

### Proof.

- ▶ While pushing from  $u$  the current-neighbour pointer is only advanced if the current edge is not admissible.
- ▶ The only thing that could make the edge admissible again would be a relabel at  $u$ .
- ▶ If we reach the end of the list ( $v = \text{null}$ ) all edges are not admissible. □

This shows that  $\text{discharge}(u)$  is correct, and that we can perform a relabel in Line 4.



## 13.2 Relabel to Front

### Algorithm 21 relabel-to-front( $G, s, t$ )

```
1: initialize preflow
2: initialize node list  $L$  containing  $V \setminus \{s, t\}$  in any order
3: foreach  $u \in V \setminus \{s, t\}$  do
4:    $u.current\text{-neighbour} \leftarrow u.neighbour\text{-list}\text{-head}$ 
5:  $u \leftarrow L.head$ 
6: while  $u \neq \text{null}$  do
7:    $old\text{-height} \leftarrow \ell(u)$ 
8:   discharge( $u$ )
9:   if  $\ell(u) > old\text{-height}$  then // relabel happened
10:    move  $u$  to the front of  $L$ 
11:    $u \leftarrow u.next$ 
```

## 13.2 Relabel to Front

### Lemma 37 (Invariant)

*In Line 6 of the relabel-to-front algorithm the following invariant holds.*

- 1. The sequence  $L$  is topologically sorted w.r.t. the set of admissible edges; this means for an admissible edge  $(x, y)$  the node  $x$  appears before  $y$  in sequence  $L$ .*
- 2. No node before  $u$  in the list  $L$  is active.*

## Proof:

### ► Initialization:

1. In the beginning  $s$  has label  $n \geq 2$ , and all other nodes have label 0. Hence, no edge is admissible, which means that any ordering  $L$  is permitted.
2. We start with  $u$  being the head of the list; hence no node before  $u$  can be active

### ► Maintenance:

1.
  - Pushes do not create any new admissible edges. Therefore, if `discharge()` does not relabel  $u$ ,  $L$  is still topologically sorted.
  - After relabeling,  $u$  cannot have admissible incoming edges as such an edge  $(x, u)$  would have had a difference  $\ell(x) - \ell(u) \geq 2$  before the re-labeling (such edges do not exist in the residual graph).  
Hence, moving  $u$  to the front does not violate the sorting property for any edge; however it fixes this property for all admissible edges leaving  $u$  that were generated by the relabeling.

## 13.2 Relabel to Front

### Proof:

► Maintenance:

2. If we do a relabel there is nothing to prove because the only node before  $u'$  ( $u$  in the next iteration) will be the current  $u$ ; the discharge( $u$ ) operation only terminates when  $u$  is not active anymore.

For the case that we do not relabel, observe that the only way a predecessor could be active is that we push flow to it via an admissible arc. However, all admissible arcs point to successors of  $u$ .

Note that the invariant means that for  $u = \text{null}$  we have a preflow with a valid labelling that does not have active nodes. This means we have a maximum flow.

## 13.2 Relabel to Front

### Lemma 38

*There are at most  $\mathcal{O}(n^3)$  calls to  $\text{discharge}(u)$ .*

Every discharge operation without a relabel advances  $u$  (the current node within list  $L$ ). Hence, if we have  $n$  discharge operations without a relabel we have  $u = \text{null}$  and the algorithm terminates.

Therefore, the number of calls to discharge is at most  $n(\#\text{relabels} + 1) = \mathcal{O}(n^3)$ .

## 13.2 Relabel to Front

### Lemma 39

*The cost for all relabel-operations is only  $\mathcal{O}(n^2)$ .*

A relabel-operation at a node is constant time (increasing the label and resetting  *$u$ .current-neighbour*). In total we have  $\mathcal{O}(n^2)$  relabel-operations.

## 13.2 Relabel to Front

Note that by definition a saturating push operation ( $\min\{c_f(e), f(u)\} = c_f(e)$ ) can at the same time be a non-saturating push operation ( $\min\{c_f(e), f(u)\} = f(u)$ ).

### Lemma 40

*The cost for all saturating push-operations that are **not** also non-saturating push-operations is only  $\mathcal{O}(mn)$ .*

Note that such a push-operation leaves the node  $u$  active but makes the edge  $e$  disappear from the residual graph. Therefore the push-operation is immediately followed by an increase of the pointer  $u.current-neighbour$ .

This pointer can traverse the neighbour-list at most  $\mathcal{O}(n)$  times (upper bound on number of relabels) and the neighbour-list has only  $degree(u) + 1$  many entries (+1 for null-entry).

## 13.2 Relabel to Front

### Lemma 41

*The cost for all non-saturating push-operations is only  $\mathcal{O}(n^3)$ .*

A non-saturating push-operation takes constant time and ends the current call to `discharge()`. Hence, there are only  $\mathcal{O}(n^3)$  such operations.

### Theorem 42

*The push-relabel algorithm with the rule relabel-to-front takes time  $\mathcal{O}(n^3)$ .*



## 13.3 Highest Label

### Algorithm 6 highest-label( $G, s, t$ )

- 1: initialize preflow
- 2: **foreach**  $u \in V \setminus \{s, t\}$  **do**
- 3:      $u.current-neighbour \leftarrow u.neighbour-list-head$
- 4: **while**  $\exists$  active node  $u$  **do**
- 5:     select active node  $u$  with highest label
- 6:     discharge( $u$ )

## 13.3 Highest Label

### Lemma 43

*When using highest label the number of non-saturating pushes is only  $\mathcal{O}(n^3)$ .*

A push from a node on level  $\ell$  can only “activate” nodes on levels strictly less than  $\ell$ .

This means, after a non-saturating push from  $u$  a relabel is required to make  $u$  active again.

Hence, after  $n$  non-saturating pushes without an intermediate relabel there are no active nodes left.

Therefore, the number of non-saturating pushes is at most  $n(\#relabels + 1) = \mathcal{O}(n^3)$ .

## 13.3 Highest Label

Since a discharge-operation is terminated by a non-saturating push this gives an upper bound of  $\mathcal{O}(n^3)$  on the number of discharge-operations.

The cost for relabels and saturating pushes can be estimated in exactly the same way as in the case of the generic push-relabel algorithm.

### Question:

How do we find the next node for a discharge operation?

## 13.3 Highest Label

Maintain lists  $L_i$ ,  $i \in \{0, \dots, 2n\}$ , where list  $L_i$  contains active nodes with label  $i$  (maintaining these lists induces only constant additional cost for every push-operation and for every relabel-operation).

After a discharge operation terminated for a node  $u$  with label  $k$ , traverse the lists  $L_k, L_{k-1}, \dots, L_0$ , (in that order) until you find a non-empty list.

Unless the last (non-saturating) push was to  $s$  or  $t$  the list  $k - 1$  must be non-empty (i.e., the search takes constant time).

## 13.3 Highest Label

Hence, the total time required for searching for active nodes is at most

$$\mathcal{O}(n^3) + n(\#non\text{-saturating-pushes-to-}s\text{-or-}t)$$

### Lemma 44

*The number of non-saturating pushes to  $s$  or  $t$  is at most  $\mathcal{O}(n^2)$ .*

With this lemma we get

### Theorem 45

*The push-relabel algorithm with the rule highest-label takes time  $\mathcal{O}(n^3)$ .*

## 13.3 Highest Label

### Proof of the Lemma.

- ▶ We only show that the number of pushes to the source is at most  $\mathcal{O}(n^2)$ . A similar argument holds for the target.
- ▶ After a node  $v$  (which must have  $\ell(v) = n + 1$ ) made a non-saturating push to the source there needs to be another node whose label is increased from  $\leq n + 1$  to  $n + 2$  before  $v$  can become active again.
- ▶ This happens for every push that  $v$  makes to the source. Since, every node can pass the threshold  $n + 2$  at most once,  $v$  can make at most  $n$  pushes to the source.
- ▶ As this holds for every node the total number of pushes to the source is at most  $\mathcal{O}(n^2)$ .

# Mincost Flow

## Problem Definition:

$$\begin{aligned} \min \quad & \sum_e c(e) f(e) \\ \text{s.t.} \quad & \forall e \in E: 0 \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{aligned}$$

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- ▶  $G = (V, E)$  is a **directed graph**.
- ▶  $u : E \rightarrow \mathbb{R}_0^+ \cup \{\infty\}$  is the **capacity function**.
- ▶  $c : E \rightarrow \mathbb{R}$  is the **cost function**  
(note that  $c(e)$  may be negative).
- ▶  $b : V \rightarrow \mathbb{R}$ ,  $\sum_{v \in V} b(v) = 0$  is a **demand function**.



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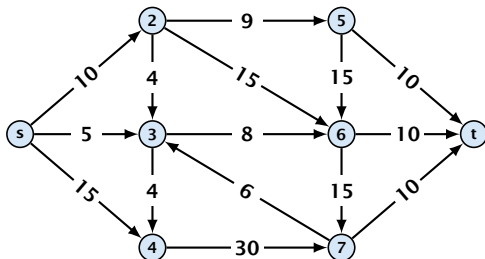
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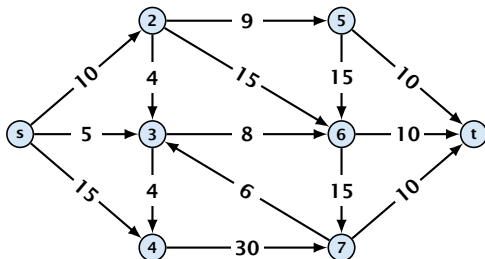
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# Solve Maxflow Using Mincost Flow

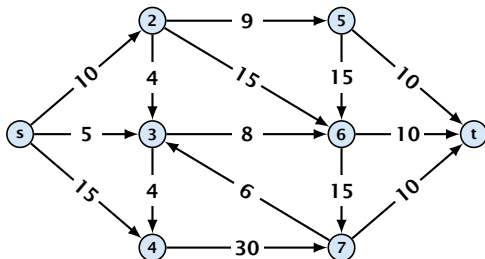


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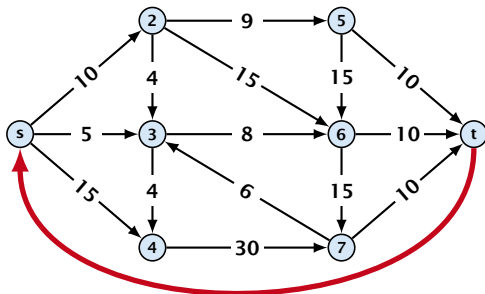
- ▶ Given a flow network for a standard maxflow problem.

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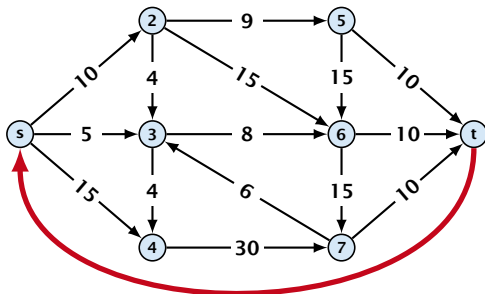
- ▶ Given a flow network for a standard maxflow problem.
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- ▶ Add an edge from  $t$  to  $s$  with infinite capacity and cost  $-1$ .
- ▶ Then,  $\text{val}(f^*) = -\text{cost}(f_{\min})$ , where  $f^*$  is a maxflow, and  $f_{\min}$  is a mincost-flow.



# Solve Maxflow Using Mincost Flow

## Solve decision version of maxflow:

- ▶ Given a flow network for a standard maxflow problem, and a value  $k$ .
- ▶ Set  $b(v) = 0$  for every node apart from  $s$  or  $t$ . Set  $b(s) = -k$  and  $b(t) = k$ .
- ▶ Set edge-costs to zero, and keep the capacities.
- ▶ There exists a maxflow of value at least  $k$  if and only if the mincost-flow problem is feasible.

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# Generalization

**Our model:**

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where  $b : V \rightarrow \mathbb{R}$ ,  $\sum_v b(v) = 0$ ;  $u : E \rightarrow \mathbb{R}_0^+ \cup \{\infty\}$ ;  $c : E \rightarrow \mathbb{R}$ ;

A more general model?

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## Differences

- ▶ Flow along an edge  $e$  may have non-zero lower bound  $\ell(e)$ .
- ▶ Flow along  $e$  may have negative upper bound  $u(e)$ .
- ▶ The demand at a node  $v$  may have lower bound  $a(v)$  and upper bound  $b(v)$  instead of just lower bound = upper bound =  $b(v)$ .

# Reduction I

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We can assume that  $a(v) = b(v)$ :

Add new nodes  $r, v$

Add new edges  $(r, v)$  for all  $v$

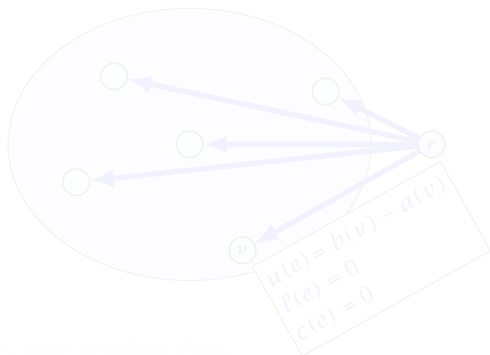
Set  $u(e) = b(v) - a(v)$  for these

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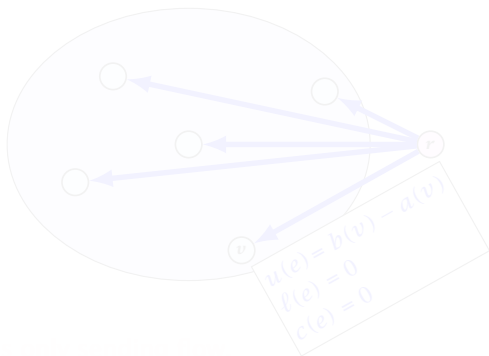




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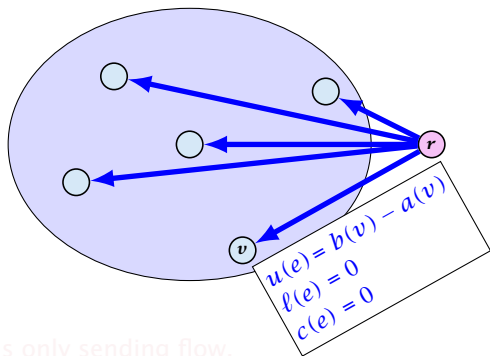
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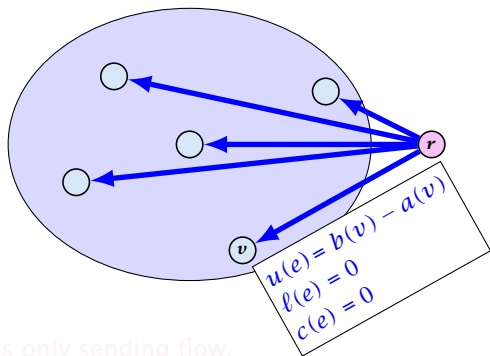
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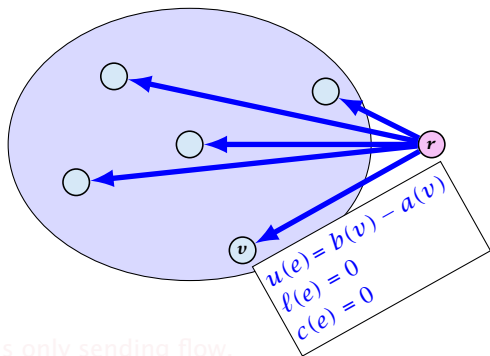
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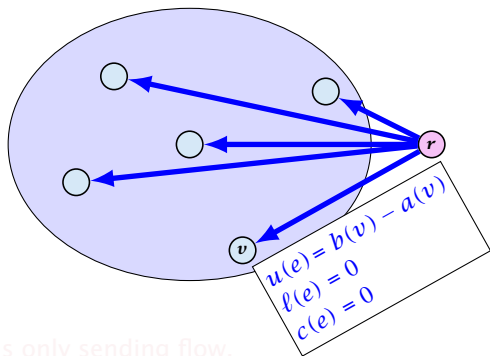
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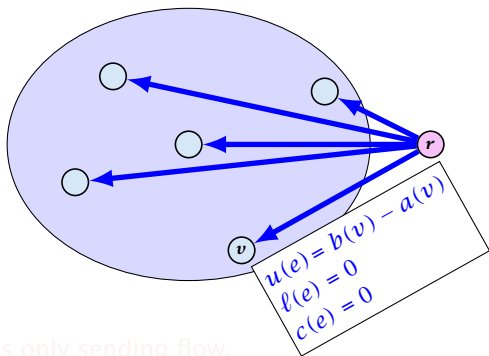
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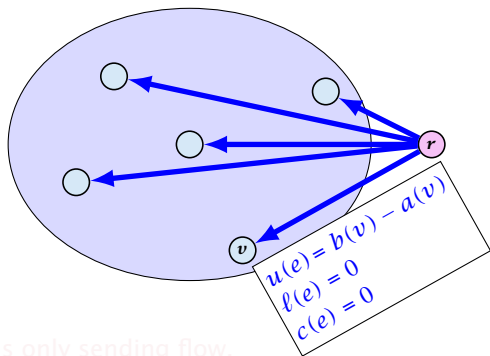
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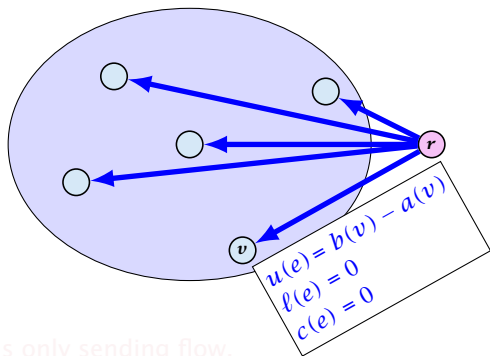
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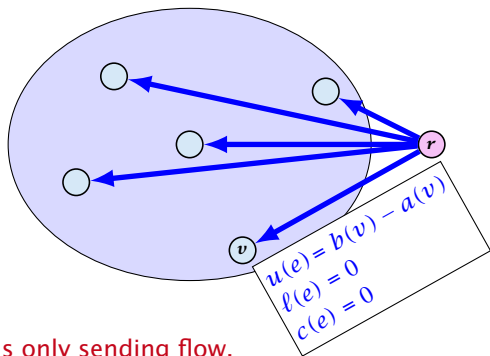
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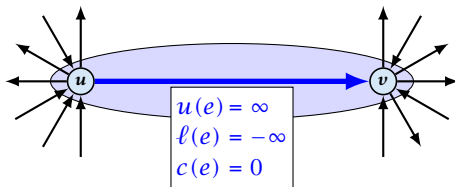
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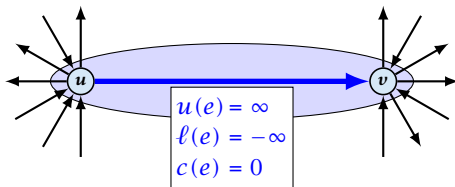
If  $c(e) = 0$  we can contract the edge/identify nodes  $u$  and  $v$ .

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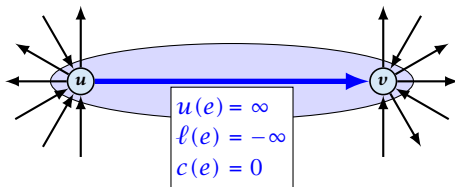
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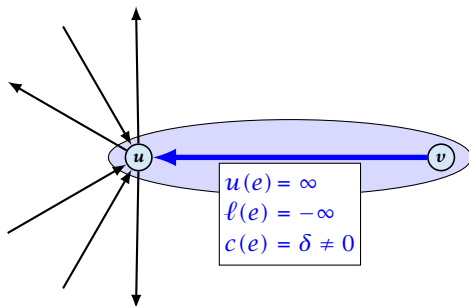


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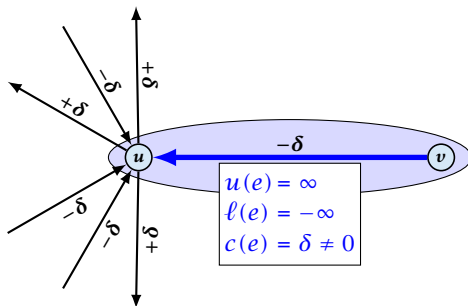
We can transform any network so that a particular edge has cost  $c(e) = 0$ :



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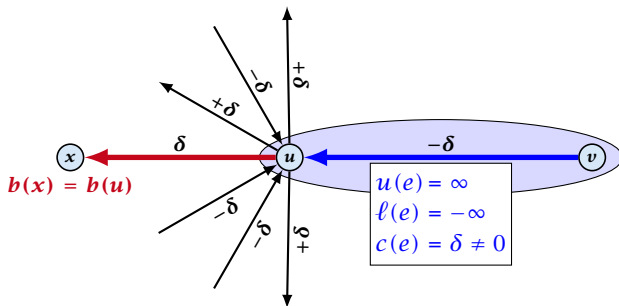
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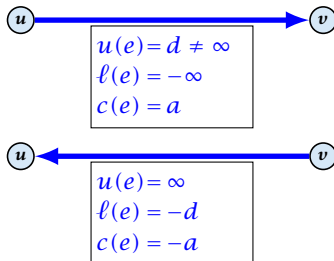


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## Reduction III

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{aligned}$$

We can assume that  $\ell(e) \neq -\infty$ :



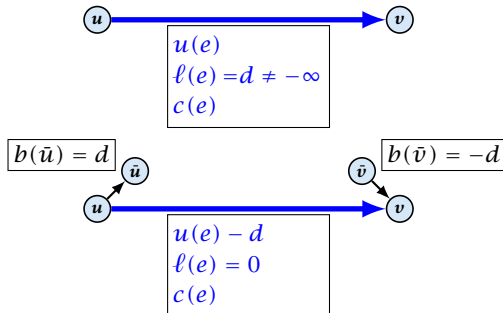
Replace the edge by an edge in opposite direction.



## Reduction IV

$$\begin{aligned} \min \quad & \sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: f(v) = b(v) \end{aligned}$$

We can assume that  $\ell(e) = 0$ :



The added edges have infinite capacity and cost  $c(e)/2$ .

## Caterer Problem

- ▶ She needs to supply  $r_i$  napkins on  $N$  successive days.
- ▶ She can buy new napkins at  $p$  cents each.
- ▶ She can launder them at a fast laundry that takes  $m$  days and cost  $f$  cents a napkin.
- ▶ She can use a slow laundry that takes  $k > m$  days and costs  $s$  cents each.
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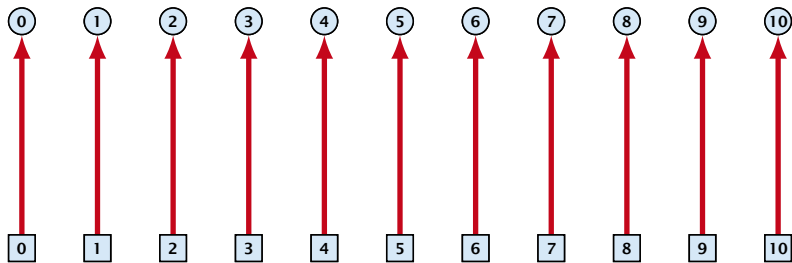
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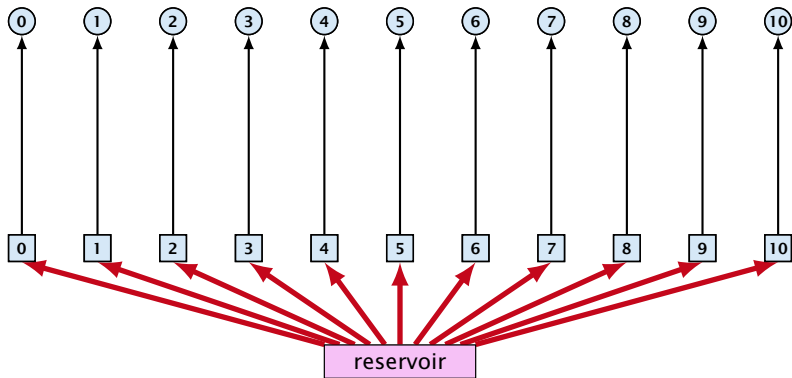






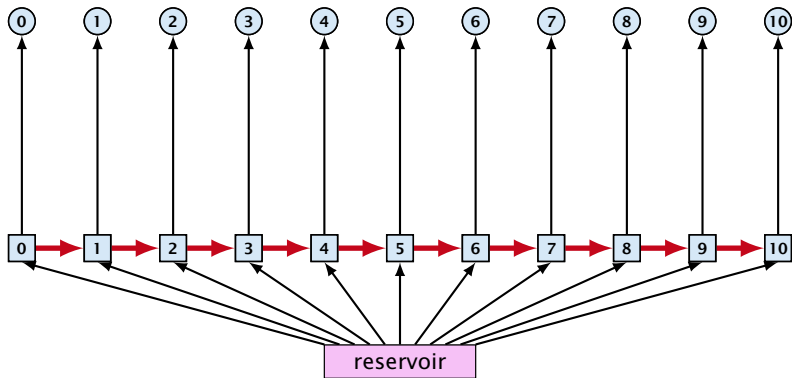
day edges:

upper bound:  $u(e_i) = \infty$ ;  
lower bound:  $\ell(e_i) = r_i$ ;  
cost:  $c(e) = 0$



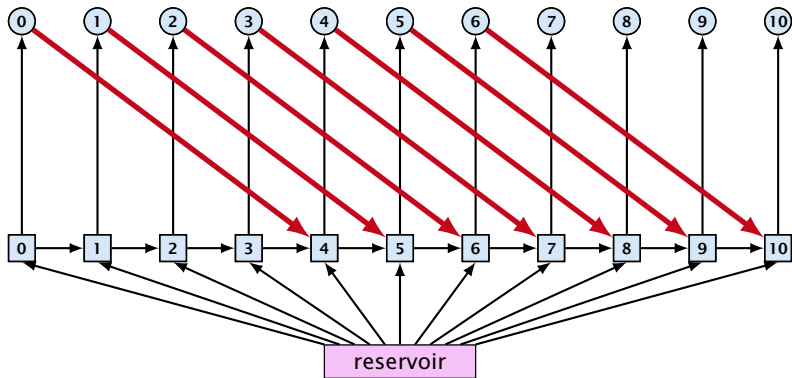
buy edges:

upper bound:  $u(e_i) = \infty$ ;  
lower bound:  $\ell(e_i) = 0$ ;  
cost:  $c(e) = p$



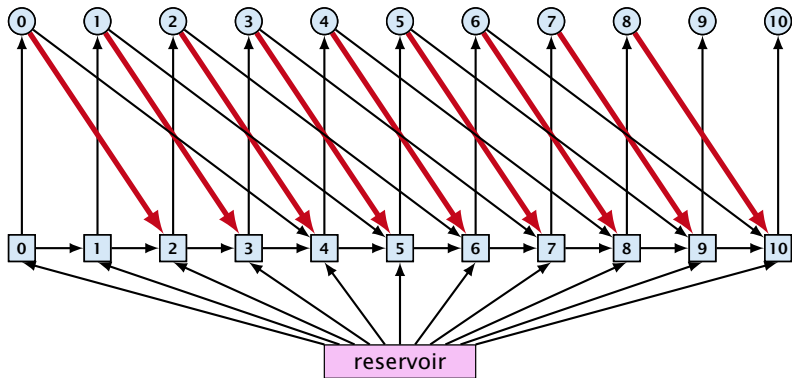
forward edges:

upper bound:  $u(e_i) = \infty$ ;  
lower bound:  $\ell(e_i) = 0$ ;  
cost:  $c(e) = 0$



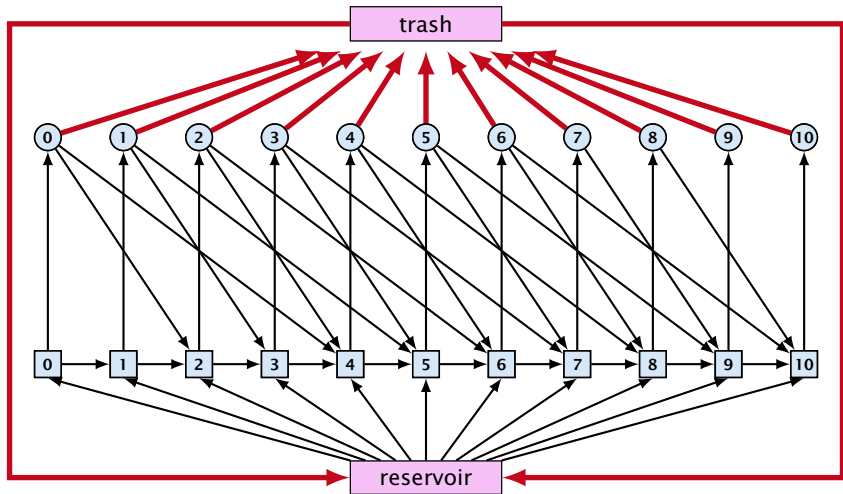
slow edges:

upper bound:  $u(e_i) = \infty$ ;  
 lower bound:  $\ell(e_i) = 0$ ;  
 cost:  $c(e) = s$



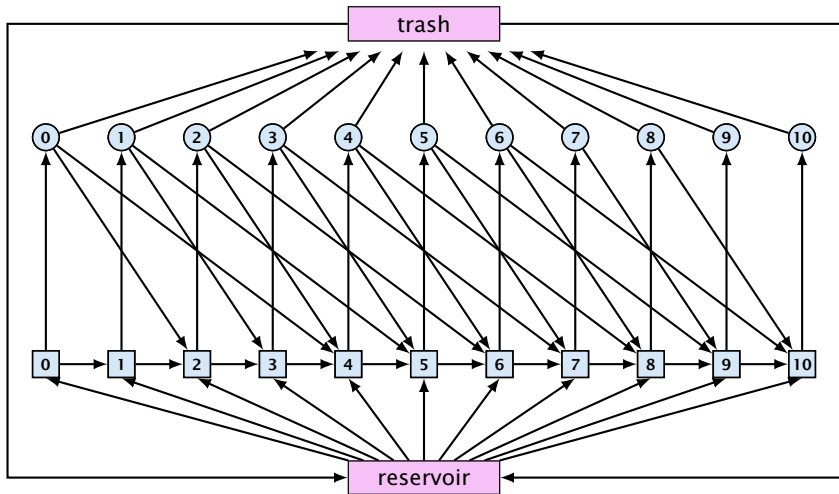
fast edges:

upper bound:  $u(e_i) = \infty$ ;  
 lower bound:  $\ell(e_i) = 0$ ;  
 cost:  $c(e) = f$



trash edges:

upper bound:  $u(e_i) = \infty$ ;  
 lower bound:  $\ell(e_i) = 0$ ;  
 cost:  $c(e) = 0$



# Residual Graph

The residual graph for a mincost flow is exactly defined as the residual graph for standard flows, with the only exception that one needs to define a cost for the residual edge.

For a flow of  $z$  from  $u$  to  $v$  the residual edge  $(v, u)$  has capacity  $z$  and a cost of  $-c((u, v))$ .



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Then  $f + g$  is a feasible flow with cost  $\text{cost}(f) + \text{cost}(g) < \text{cost}(f)$ . Hence,  $f$  is not minimum cost.

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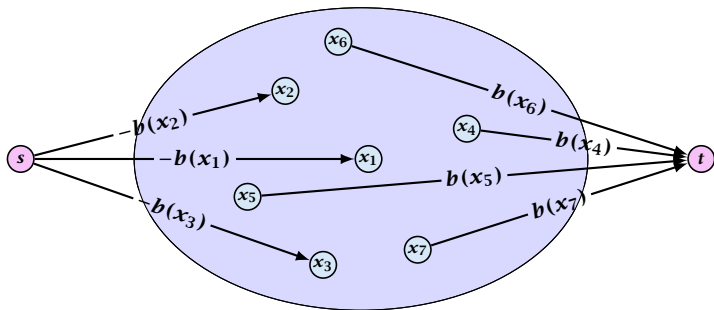
# 14 Mincost Flow

## Algorithm 22 CycleCanceling( $G = (V, E), c, u, b$ )

- 1: establish a feasible flow  $f$  in  $G$
- 2: **while**  $G_f$  contains negative cycle **do**
- 3:     use Bellman-Ford to find a negative circuit  $Z$
- 4:      $\delta \leftarrow \min\{u_f(e) \mid e \in Z\}$
- 5:     augment  $\delta$  units along  $Z$  and update  $G_f$



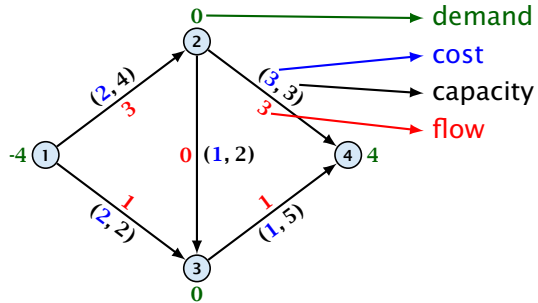
## How do we find the initial feasible flow?



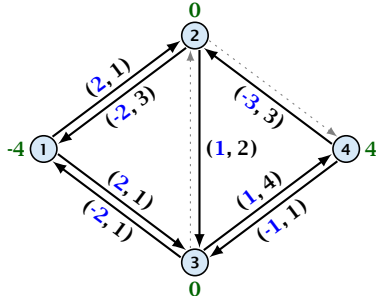
- ▶ Connect new node  $s$  to all nodes with negative  $b(v)$ -value.
- ▶ Connect nodes with positive  $b(v)$ -value to a new node  $t$ .
- ▶ There exist a feasible flow in the original graph iff in the resulting graph there exists an  $s$ - $t$  flow of value

$$\sum_{v:b(v)<0} (-b(v)) = \sum_{v:b(v)>0} b(v) .$$

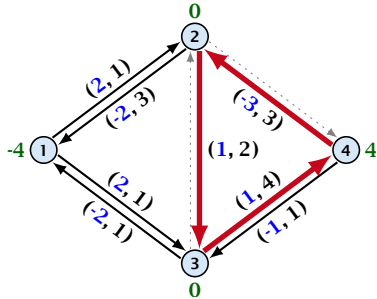
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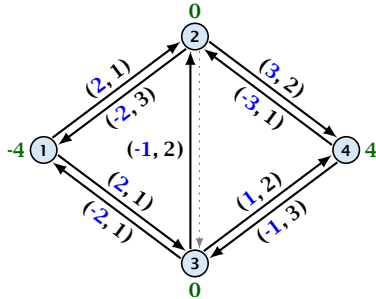
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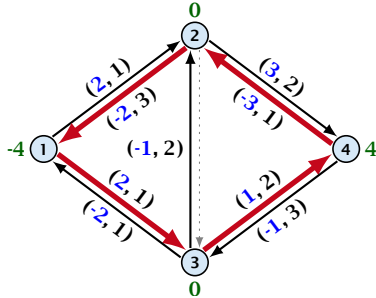
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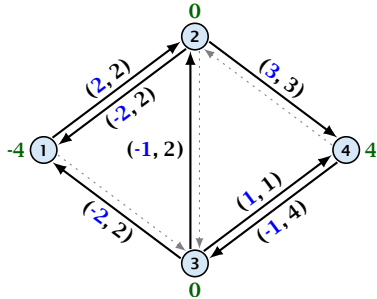
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## Lemma 48

The improving cycle algorithm runs in time  $\mathcal{O}(nm^2CU)$ , for integer capacities and costs, when for all edges  $e$ ,  $|c(e)| \leq C$  and  $|u(e)| \leq U$ .

- ▶ Running time of Bellman-Ford is  $\mathcal{O}(mn)$ .
- ▶ Pushing flow along the cycle can be done in time  $\mathcal{O}(n)$ .
- ▶ Each iteration decreases the total cost by at least 1.
- ▶ The true optimum cost must lie in the interval  $[-mCU, \dots, +mCU]$ .

Note that this lemma is weak since it does not allow for edges with infinite capacity.



# 14 Mincost Flow

A **general mincost flow problem** is of the following form:

$$\begin{aligned} \min \quad & \sum_e c(e) f(e) \\ \text{s.t.} \quad & \forall e \in E: \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: a(v) \leq f(v) \leq b(v) \end{aligned}$$

where  $a: V \rightarrow \mathbb{R}$ ,  $b: V \rightarrow \mathbb{R}$ ;  $\ell: E \rightarrow \mathbb{R} \cup \{-\infty\}$ ,  $u: E \rightarrow \mathbb{R} \cup \{\infty\}$   
 $c: E \rightarrow \mathbb{R}$ ;

## Lemma 49 (without proof)

*A general mincost flow problem can be solved in polynomial time.*