Let $I$ denote the solution obtained by the first rounding algorithm and $I'$ be the solution returned by the second algorithm. Then

$$I \subseteq I' \ .$$

This means $I'$ is never better than $I$.

- Suppose that we take $S_i$ in the first algorithm. I.e., $i \in I$.
- This means $x_i \geq \frac{1}{f}$.
- Because of Complementary Slackness Conditions the corresponding constraint in the dual must be tight.
- Hence, the second algorithm will also choose $S_i$.



## Technique 3: The Primal Dual Method

The previous two rounding algorithms have the disadvantage that it is necessary to solve the LP. The following method also gives an $f$-approximation without solving the LP.

For estimating the cost of the solution we only required two properties.

1. The solution is dual feasible and, hence,

$$\sum_u y_u \leq \text{cost}(x^*) \leq \text{OPT}$$

   where $x^*$ is an optimum solution to the primal LP.
2. The set $I$ contains only sets for which the dual inequality is tight.

Of course, we also need that $I$ is a cover.



## Technique 3: The Primal Dual Method

**Algorithm 1** PrimalDual

```
1: y ← 0
2: I ← ∅
3: while exists u ∉ ⋃_{i∈I} S_i do
4:     increase dual variable y_u until constraint for some
       new set S_ℓ becomes tight
5:     I ← I ∪ {ℓ}
```



## Technique 4: The Greedy Algorithm

**Algorithm 1** Greedy

```
1: I ← ∅
2: Ŝ_j ← S_j    for all j
3: while I not a set cover do
4:     ℓ ← arg min_{j:Ŝ_j≠0} w_j/|Ŝ_j|
5:     I ← I ∪ {ℓ}
6:     Ŝ_j ← Ŝ_j − S_ℓ    for all j
```

In every round the Greedy algorithm takes the set that covers remaining elements in the most cost-effective way.

We choose a set such that the ratio between cost and still uncovered elements in the set is minimized.

## Technique 4: The Greedy Algorithm

**Lemma 4**

*Given positive numbers* $a_1, \ldots, a_k$ *and* $b_1, \ldots, b_k$*, and* $S \subseteq \{1, \ldots, k\}$ *then*

$$\min_i \frac{a_i}{b_i} \le \frac{\sum_{i \in S} a_i}{\sum_{i \in S} b_i} \le \max_i \frac{a_i}{b_i}$$



## Technique 4: The Greedy Algorithm

Let $n_\ell$ denote the number of elements that remain at the beginning of iteration $\ell$. $n_1 = n = |U|$ and $n_{s+1} = 0$ if we need $s$ iterations.

In the $\ell$-th iteration

$$\min_j \frac{w_j}{|\hat{S}_j|} \le \frac{\sum_{j \in \mathrm{OPT}} w_j}{\sum_{j \in \mathrm{OPT}} |\hat{S}_j|} = \frac{\mathrm{OPT}}{\sum_{j \in \mathrm{OPT}} |\hat{S}_j|} \le \frac{\mathrm{OPT}}{n_\ell}$$

since an optimal algorithm can cover the remaining $n_\ell$ elements with cost $\mathrm{OPT}$.

Let $\hat{S}_j$ be a subset that minimizes this ratio. Hence, $w_j / |\hat{S}_j| \le \frac{\mathrm{OPT}}{n_\ell}$.



## Technique 4: The Greedy Algorithm

Adding this set to our solution means $n_{\ell+1} = n_\ell - |\hat{S}_j|$.

$$w_j \le \frac{|\hat{S}_j| \mathrm{OPT}}{n_\ell} = \frac{n_\ell - n_{\ell+1}}{n_\ell} \cdot \mathrm{OPT}$$

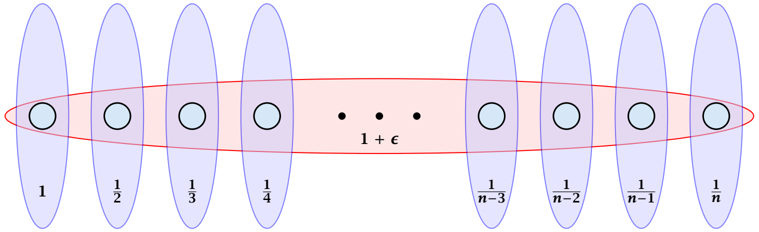


## Technique 4: The Greedy Algorithm

$$\begin{aligned}
\sum_{j \in I} w_j &\le \sum_{\ell=1}^{s} \frac{n_\ell - n_{\ell+1}}{n_\ell} \cdot \mathrm{OPT} \\
&\le \mathrm{OPT} \sum_{\ell=1}^{s} \left( \frac{1}{n_\ell} + \frac{1}{n_\ell - 1} + \cdots + \frac{1}{n_{\ell+1} + 1} \right) \\
&= \mathrm{OPT} \sum_{i=1}^{k} \frac{1}{i} \\
&= H_n \cdot \mathrm{OPT} \le \mathrm{OPT}(\ln n + 1) \ .
\end{aligned}$$

## Technique 4: The Greedy Algorithm

**A tight example:**






## Technique 5: Randomized Rounding

One round of randomized rounding:
Pick set $S_j$ uniformly at random with probability $1 - x_j$ (for all $j$).

**Version A:** Repeat rounds until you have a cover.

**Version B:** Repeat for $s$ rounds. If you have a cover STOP.
Otherwise, repeat the whole algorithm.



**Probability that $u \in U$ is not covered (in one round):**

$$\begin{aligned}\Pr[u \text{ not covered in one round}] &= \prod_{j:u\in S_j}(1-x_j) \le \prod_{j:u\in S_j} e^{-x_j}\\ &= e^{-\sum_{j:u\in S_j} x_j} \le e^{-1}\ .\end{aligned}$$

**Probability that $u \in U$ is not covered (after $\ell$ rounds):**

$$\Pr[u \text{ not covered after } \ell \text{ round}] \le \frac{1}{e^{\ell}}\ .$$



$$\begin{aligned}&\Pr[\exists u \in U \text{ not covered after } \ell \text{ round}]\\ &\quad = \Pr[u_1 \text{ not covered} \vee u_2 \text{ not covered} \vee \ldots \vee u_n \text{ not covered}]\\ &\quad \le \sum_i \Pr[u_i \text{ not covered after } \ell \text{ rounds}] \le ne^{-\ell}\ .\end{aligned}$$

### Lemma 5

*With high probability $\mathcal{O}(\log n)$ rounds suffice.*

**With high probability:**
For any constant $\alpha$ the number of rounds is at most $\mathcal{O}(\log n)$ with probability at least $1 - n^{-\alpha}$.