Part III

Approximation Algorithms

There are many practically important optimization problems that are NP-hard.

What can we do?

- Heuristics.
- Exploit special structure of instances occurring in practice.
- Consider algorithms that do not compute the optimal solution but provide solutions that are close to optimum.

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Definition 2

An α-approximation for an optimization problem is a polynomial-time algorithm that for all instances of the problem produces a solution whose value is within a factor of α of the value of an optimal solution.

Why approximation algorithms?

- We need algorithms for hard problems.
- It gives a rigorous mathematical base for studying heuristics.
- It provides a metric to compare the difficulty of various optimization problems.
- Proving theorems may give a deeper theoretical understanding which in turn leads to new algorithmic approaches.

Why not?

- Sometimes the results are very pessimistic due to the fact that an algorithm has to provide a close-to-optimum solution on every instance.

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Definition 3

An optimization problem $P = (\mathcal{I}, \mathrm{sol}, m, \mathrm{goal})$ is in **NPO** if

- $x \in \mathcal{I}$ can be decided in polynomial time
- $y \in \mathrm{sol}(\mathcal{I})$ can be verified in polynomial time
- m can be computed in polynomial time
- $\mathrm{goal} \in \{\min, \max\}$

In other words: the decision problem is there a solution y with $m(x, y)$ at most/at least z is in NP.

- x is problem instance
- y is candidate solution
- $m^*(x)$ cost/profit of an optimal solution

Definition 4 (Performance Ratio)

$$R(x,y) := \max\left\{\frac{m(x,y)}{m^*(x)}, \frac{m^*(x)}{m(x,y)}\right\}$$

Definition 5 (r-approximation)

An algorithm A is an r-approximation algorithm iff

$$\forall x \in \mathcal{I} : R(x, A(x)) \leq r \quad ,$$

and A runs in polynomial time.

Definition 6 (PTAS)
A PTAS for a problem P from NPO is an algorithm that takes as input $x \in \mathcal{I}$ and $\epsilon > 0$ and produces a solution y for x with

$$R(x, y) \leq 1 + \epsilon \enspace .$$

The running time is polynomial in $|x|$.

approximation with arbitrary good factor... fast?

Problems that have a PTAS

Scheduling. Given m jobs with known processing times; schedule the jobs on n machines such that the MAKESPAN is minimized.

Definition 7 (FPTAS)

An FPTAS for a problem P from NPO is an algorithm that takes as input $x \in \mathcal{I}$ and $\epsilon > 0$ and produces a solution y for x with

$$R(x,y) \leq 1+\epsilon \ .$$

The running time is polynomial in $|x|$ and $1/\epsilon$.

approximation with arbitrary good factor... fast!

Problems that have an FPTAS

KNAPSACK. Given a set of items with profits and weights choose a subset of total weight at most W s.t. the profit is maximized.

Definition 8 (APX – approximable)

A problem P from NPO is in APX if there exist a constant $r \geq 1$ and an r-approximation algorithm for P.

constant factor approximation...

Problems that are in APX

MAXCUT. Given a graph $G = (V, E)$; partition V into two disjoint pieces A and B s. t. the number of edges between both pieces is maximized.

MAX-3SAT. Given a 3CNF-formula. Find an assignment to the variables that satisfies the maximum number of clauses.

Problems with polylogarithmic approximation guarantees

- Set Cover
- Minimum Multicut
- Sparsest Cut
- Minimum Bisection

There is an r-approximation with $r \leq \mathcal{O}(\log^c(|x|))$ for some constant c.

Note that only for some of the above problem a matching lower bound is known.

There are really difficult problems!

Theorem 9
For any constant $\epsilon > 0$ there does not exist an $\Omega(n^{1-\epsilon})$-approximation algorithm for the maximum clique problem on a given graph G with n nodes unless $\mathrm{P} = \mathrm{NP}$.

Note that an n-approximation is trivial.

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There are weird problems!

Asymmetric k-Center admits an $\mathcal{O}(\log^* n)$-approximation.

There is no $o(\log^* n)$-approximation to Asymmetric k-Center unless $NP \subseteq DTIME(n^{\log\log\log n})$.

Class APX not important in practise.

Instead of saying problem P is in APX one says problem P admits a 4-approximation.

One only says that a problem is APX-hard.

A crucial ingredient for the design and analysis of approximation algorithms is a technique to obtain an upper bound (for maximization problems) or a lower bound (for minimization problems).

Therefore Linear Programs or Integer Linear Programs play a vital role in the design of many approximation algorithms.

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An Integer Linear Program or Integer Program is a Linear Program in which all variables are required to be integral.

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Set Cover

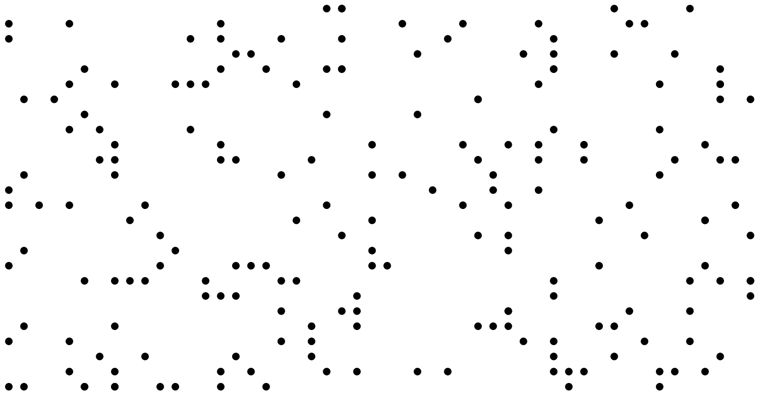
Given a ground set U, a collection of subsets $S_1, \dots, S_k \subseteq U$, where the i-th subset S_i has weight/cost w_i. Find a collection $I \subseteq \{1, \dots, k\}$ such that

$$\forall u \in U \exists i \in I : \ u \in S_i \quad \text{(every element is covered)}$$

and

$$\sum_{i \in I} w_i \quad \text{is minimized.}$$

Set Cover



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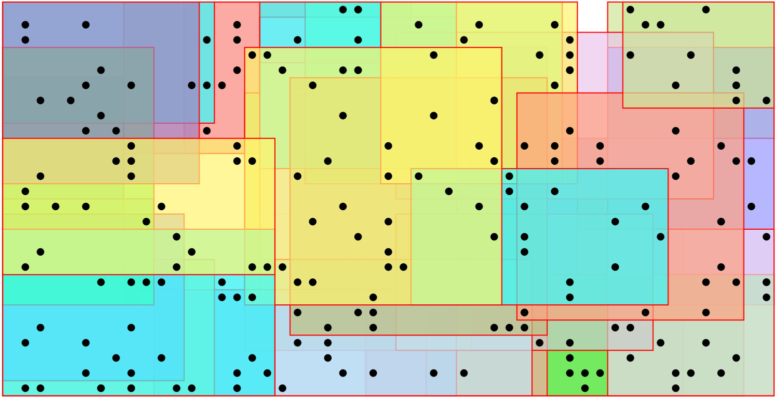

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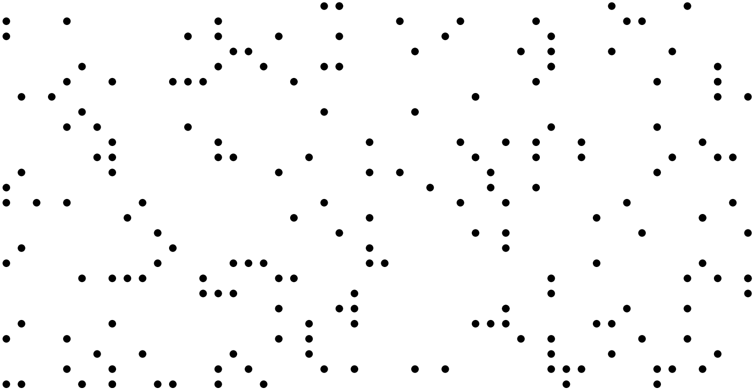

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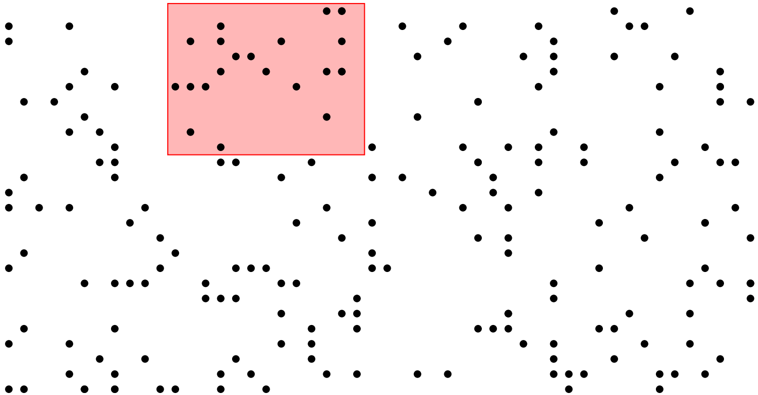

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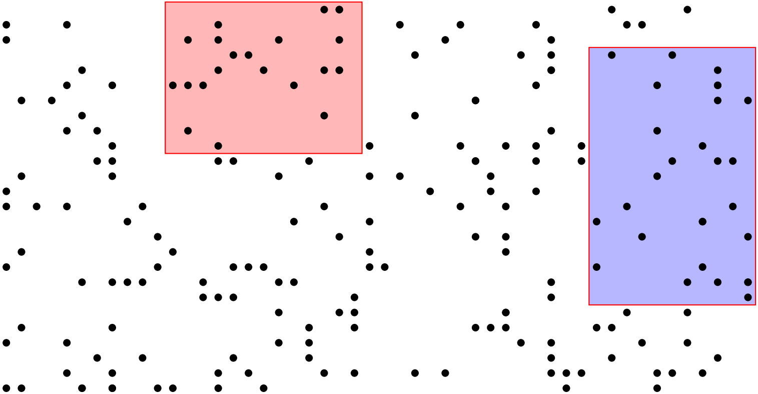

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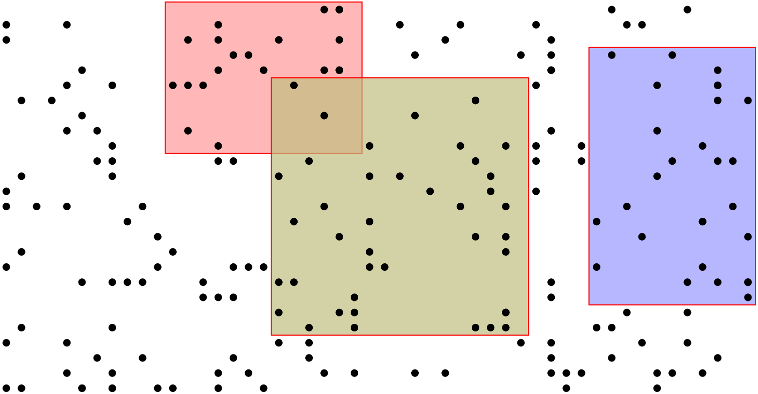

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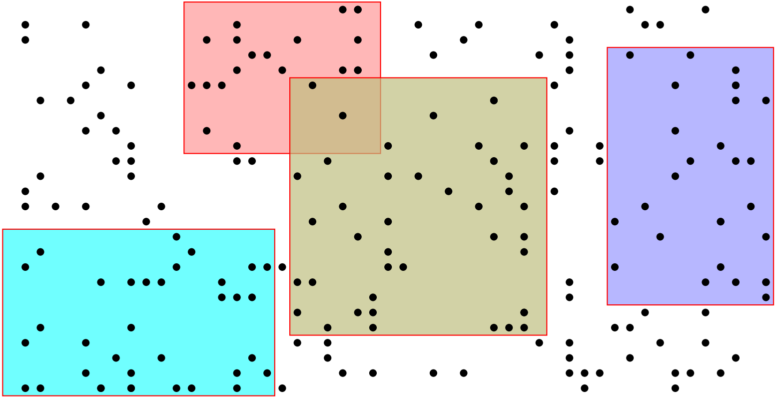

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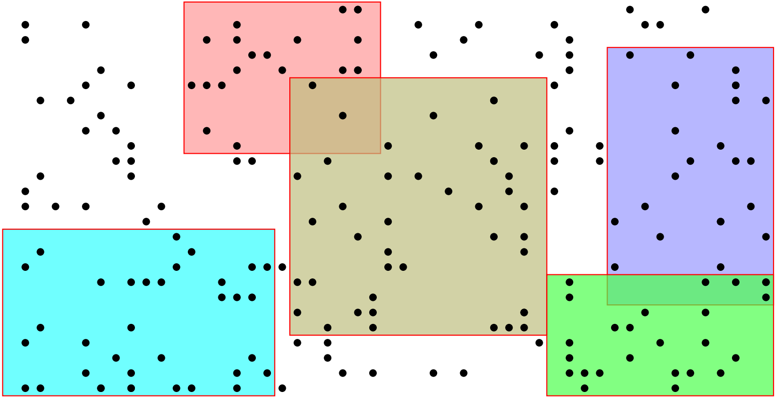

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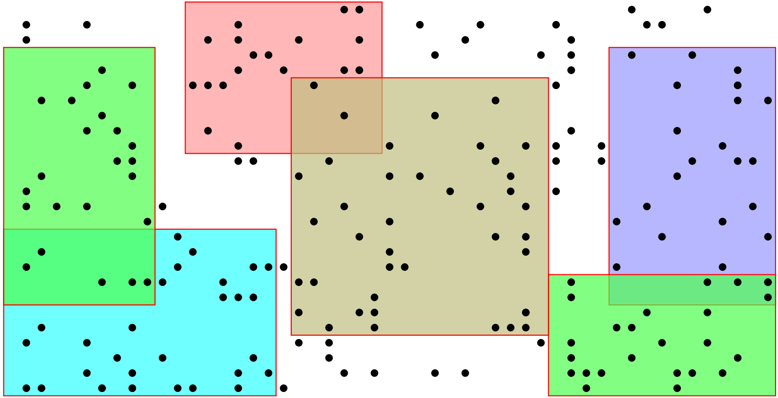

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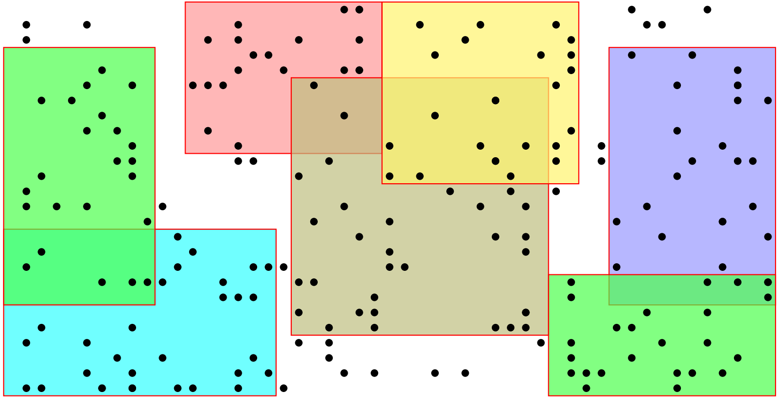

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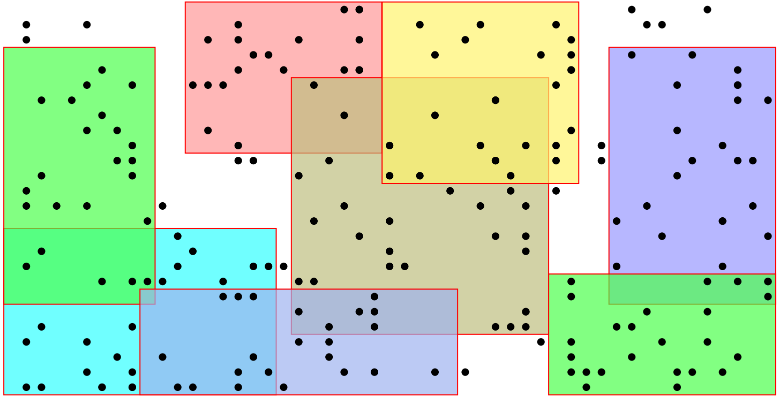

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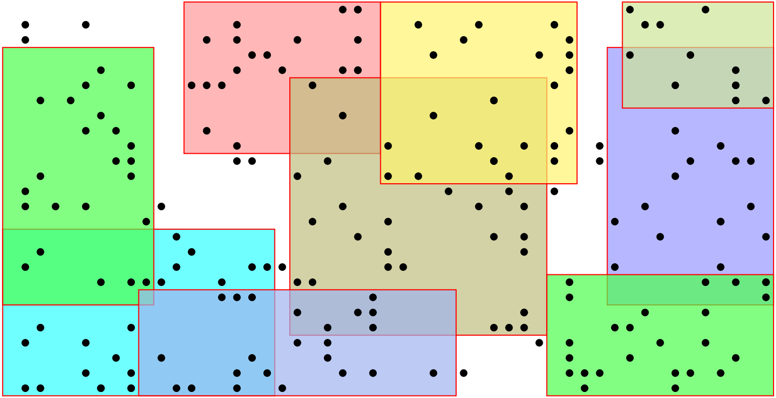

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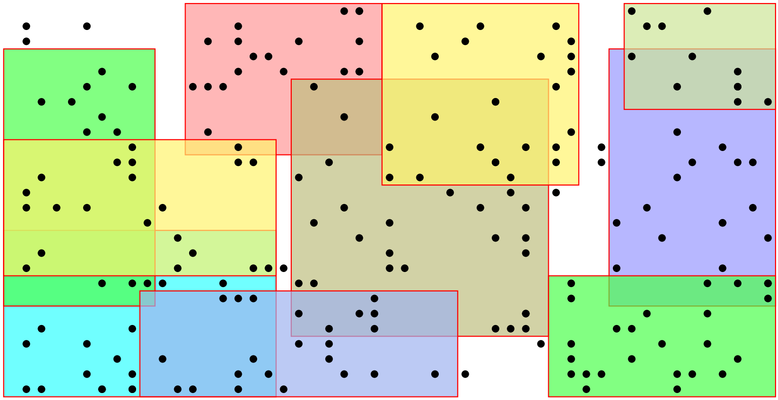

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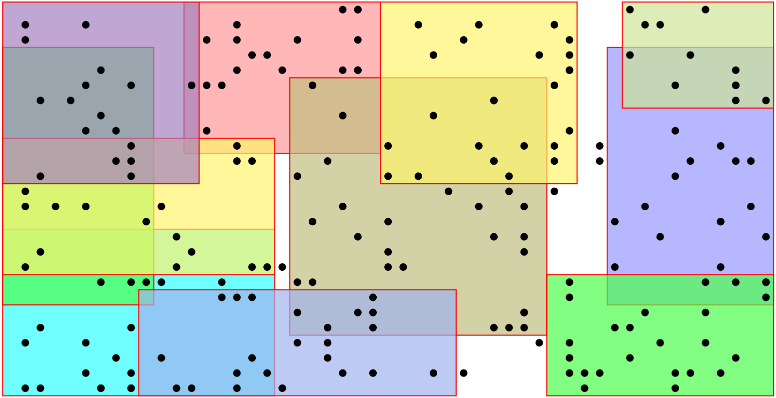

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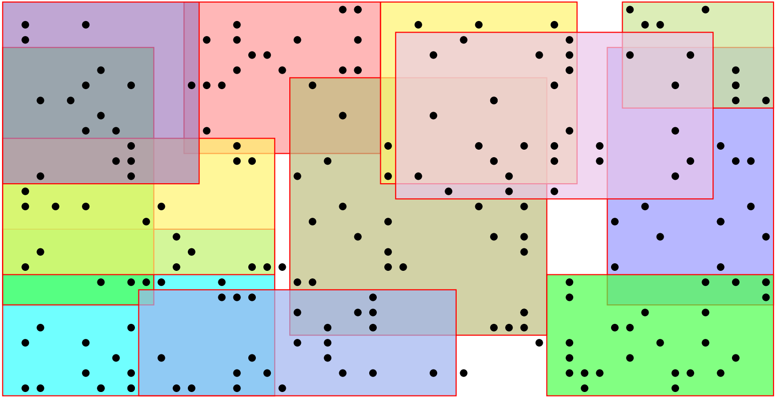

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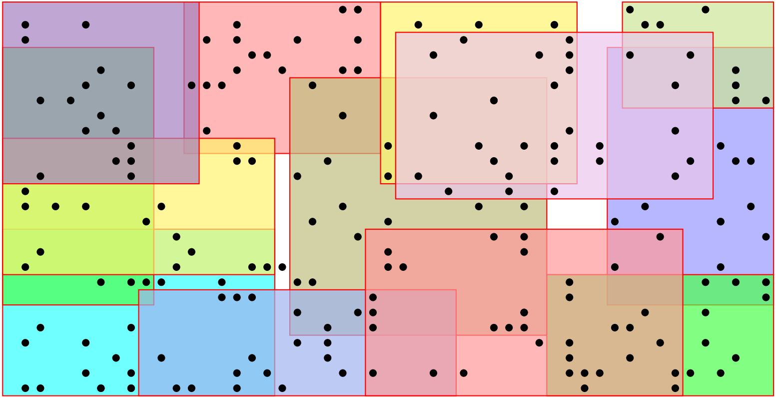

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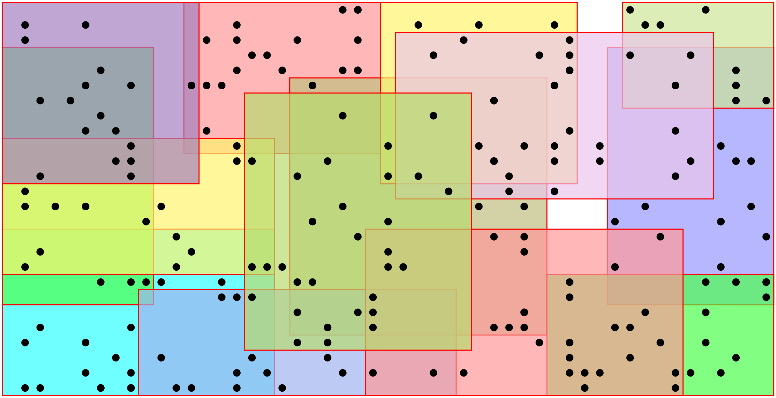

IP-Formulation of Set Cover

$$
\begin{array}{llrcl}
\min & & \sum_i w_i x_i & & \\
\text{s.t.} & \forall u \in U & \sum_{i:u\in S_i} x_i & \geq & 1 \\
& \forall i \in \{1,\dots,k\} & x_i & \geq & 0 \\
& \forall i \in \{1,\dots,k\} & x_i & \text{integral} &
\end{array}
$$

Vertex Cover

Given a graph $G = (V, E)$ and a weight w_v for every node. Find a vertex subset $S \subseteq V$ of minimum weight such that every edge is incident to at least one vertex in S.

IP-Formulation of Vertex Cover

$$
\begin{array}{llrcl}
\min & & \sum_{v\in V} w_v x_v & & \\
\text{s.t.} & \forall e=(i,j)\in E & x_i + x_j & \geq & 1 \\
 & \forall v\in V & x_v & \in & \{0,1\}
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Maximum Weighted Matching

Given a graph $G = (V, E)$, and a weight w_e for every edge $e \in E$. Find a subset of edges of maximum weight such that no vertex is incident to more than one edge.

$$\begin{array}{llrcl} \max & & \sum_{e \in E} w_e x_e & & \\ \text{s.t.} & \forall v \in V & \sum_{e: v \in e} x_e & \leq & 1 \\ & \forall e \in E & x_e & \in & \{0, 1\} \end{array}$$

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Maximum Independent Set

Given a graph $G = (V, E)$, and a weight w_v for every node $v \in V$.
Find a subset $S \subseteq V$ of nodes of maximum weight such that no two vertices in S are adjacent.

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Knapsack

Given a set of items $\{1, \dots, n\}$, where the i-th item has weight w_i and profit p_i, and given a threshold K. Find a subset $I \subseteq \{1, \dots, n\}$ of items of total weight at most K such that the profit is maximized.

$$\begin{array}{llrcl} \max & & \sum_{i=1}^{n} p_i x_i & & \\ \text{s.t.} & & \sum_{i=1}^{n} w_i x_i & \leq & K \\ & \forall i \in \{1, \dots, n\} & x_i & \in & \{0, 1\} \end{array}$$

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Relaxations

Definition 12
A linear program LP is a relaxation of an integer program IP if any feasible solution for IP is also feasible for LP and if the objective values of these solutions are identical in both programs.

We obtain a relaxation for all examples by writing $x_i \in [0,1]$ instead of $x_i \in \{0,1\}$.

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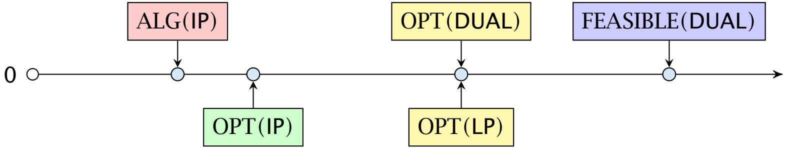
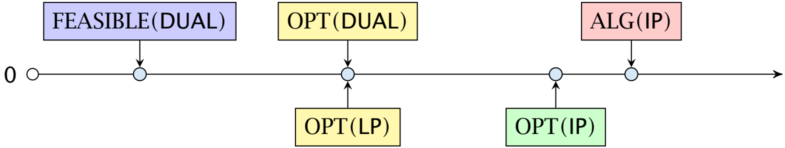
By solving a relaxation we obtain an upper bound for a maximization problem and a lower bound for a minimization problem.

Relations

Maximization Problems:



Minimization Problems:



Technique 1: Round the LP solution.

We first solve the LP-relaxation and then we round the fractional values so that we obtain an integral solution.

Set Cover relaxation:

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Let f_u be the number of sets that the element u is contained in (the frequency of u). Let $f = \max_u \{f_u\}$ be the maximum frequency.

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Rounding Algorithm:

Set all x_i-values with $x_i \geq \frac{1}{f}$ to 1. Set all other x_i-values to 0.

Technique 1: Round the LP solution.

Lemma 13

The rounding algorithm gives an f-approximation.

Proof: Every $u \in U$ is covered.

- We know that $\sum_{i: u \in S_i} x_i \geq 1$.
- The sum contains at most $f_u \leq f$ elements.
- Therefore one of the sets that contain u must have $x_i \geq 1/f$.
- This set will be selected. Hence, u is covered.

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$$\sum_{i \in I} w_i \le \sum_{i=1}^{k} w_i (f \cdot x_i)$$

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$$\sum_{i \in I} w_i \le \sum_{i=1}^{k} w_i (f \cdot x_i) = f \cdot \mathrm{cost}(x)$$

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$$\begin{aligned}\sum_{i\in I} w_i &\le \sum_{i=1}^{k} w_i(f\cdot x_i)\\ &= f\cdot \mathrm{cost}(x)\\ &\le f\cdot \mathrm{OPT}\enspace .\end{aligned}$$

Technique 2: Rounding the Dual Solution.

Relaxation for Set Cover

Primal:

$$\begin{array}{lll} \min & \sum_{i\in I} w_i x_i & \\ \text{s.t. } \forall u & \sum_{i:u\in S_i} x_i & \geq 1 \\ & x_i & \geq 0 \end{array}$$

Dual:

$$\begin{array}{lll} \max & \sum_{u\in U} y_u & \\ \text{s.t. } \forall i & \sum_{u:u\in S_i} y_u & \leq w_i \\ & y_u & \geq 0 \end{array}$$

Technique 2: Rounding the Dual Solution.

Relaxation for Set Cover

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$$\begin{array}{lll} \min & \sum_{i\in I} w_i x_i & \\ \text{s.t. } \forall u & \sum_{i:u\in S_i} x_i & \geq 1 \\ & x_i & \geq 0 \end{array}$$

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$$\begin{array}{lll} \max & \sum_{u\in U} y_u & \\ \text{s.t. } \forall i & \sum_{u:u\in S_i} y_u & \leq w_i \\ & y_u & \geq 0 \end{array}$$

Technique 2: Rounding the Dual Solution.

Relaxation for Set Cover

Primal:

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Technique 2: Rounding the Dual Solution.

Rounding Algorithm:

Let I denote the index set of sets for which the dual constraint is tight. This means for all $i \in I$

$$\sum_{u: u \in S_i} y_u = w_i$$

Technique 2: Rounding the Dual Solution.

Lemma 14

The resulting index set is an f-approximation.

Proof:

Every $u \in U$ is covered.

- Suppose there is a u that is not covered.
- This means $\sum_{u:u\in S_j} y_u < w_j$ for all sets S_j that contain u.
- But then y_u could be increased in the dual solution without violating any constraint. This is a contradiction to the fact that the dual solution is optimal.

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Technique 2: Rounding the Dual Solution.

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$$\begin{aligned}\sum_{i\in I} w_i &= \sum_{i\in I}\sum_{u:u\in S_i} y_u\\ &= \sum_u |\{i\in I : u\in S_i\}|\cdot y_u\\ &\le \sum_u f_u y_u\end{aligned}$$

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Let I denote the solution obtained by the first rounding algorithm and I' be the solution returned by the second algorithm. Then

$$I \subseteq I' \ .$$

This means I' is never better than I.

- Suppose that we take S_j in the first algorithm. I.e., $j \in I$.
- This means $x_j^* \geq \frac{1}{f}$.
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Technique 3: The Primal Dual Method

The previous two rounding algorithms have the disadvantage that it is necessary to solve the LP. The following method also gives an f-approximation without solving the LP.

For estimating the cost of the solution we only required two properties.

- The solution is dual feasible and, hence,

$$\sum_e y_e \le \text{cost}(x^*) \le \text{OPT}$$

  where x^* is an optimum solution to the primal LP.
- The set I contains only sets for which the dual inequality is tight.

Of course, we also need that I is a cover.

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Technique 3: The Primal Dual Method

Algorithm 1 PrimalDual

1: $y \leftarrow 0$
2: $I \leftarrow \emptyset$
3: **while** exists $u \notin \bigcup_{i \in I} S_i$ **do**
4: increase dual variable y_u until constraint for some new set S_ℓ becomes tight
5: $I \leftarrow I \cup \{\ell\}$

Technique 4: The Greedy Algorithm

Algorithm 1 Greedy

1: $I \leftarrow \emptyset$
2: $\hat{S}_j \leftarrow S_j$ for all j
3: **while** I not a set cover **do**
4: $\quad \ell \leftarrow \arg\min_{j:\hat{S}_j \neq 0} \frac{w_j}{|\hat{S}_j|}$
5: $\quad I \leftarrow I \cup \{\ell\}$
6: $\quad \hat{S}_j \leftarrow \hat{S}_j - S_\ell$ for all j

In every round the Greedy algorithm takes the set that covers remaining elements in the most cost-effective way.

We choose a set such that the ratio between cost and still uncovered elements in the set is minimized.

Technique 4: The Greedy Algorithm

Lemma 15

Given positive numbers $a_1, \dots, a_k$ *and* $b_1, \dots, b_k$*, and* $S \subseteq \{1, \dots, k\}$ *then*

$$\min_i \frac{a_i}{b_i} \le \frac{\sum_{i \in S} a_i}{\sum_{i \in S} b_i} \le \max_i \frac{a_i}{b_i}$$

Technique 4: The Greedy Algorithm

Let n_ℓ denote the number of elements that remain at the beginning of iteration ℓ. $n_1 = n = |U|$ and $n_{s+1} = 0$ if we need s iterations.

In the ℓ-th iteration

$$\min_j \frac{w_j}{|\hat{S}_j|} \le \frac{\sum_{j\in\mathrm{OPT}} w_j}{\sum_{j\in\mathrm{OPT}} |\hat{S}_j|} = \frac{\mathrm{OPT}}{\sum_{j\in\mathrm{OPT}} |\hat{S}_j|} \le \frac{\mathrm{OPT}}{n_\ell}$$

since an optimal algorithm can cover the remaining n_ℓ elements with cost OPT.

Let $\hat{S}_j$ be a subset that minimizes this ratio. Hence, $w_j/|\hat{S}_j| \le \frac{\mathrm{OPT}}{n_\ell}$.

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Adding this set to our solution means $n_{\ell+1} = n_\ell - |\hat{S}_j|$.

$$w_j \le \frac{|\hat{S}_j|\mathrm{OPT}}{n_\ell} = \frac{n_\ell - n_{\ell+1}}{n_\ell} \cdot \mathrm{OPT}$$

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$$\sum_{j \in I} w_j$$

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$$\sum_{j\in I} w_j \leq \sum_{\ell=1}^{s} \frac{n_\ell - n_{\ell+1}}{n_\ell} \cdot \mathrm{OPT}$$

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$$
\begin{aligned}
\sum_{j\in I} w_j &\le \sum_{\ell=1}^{s} \frac{n_\ell - n_{\ell+1}}{n_\ell} \cdot \mathrm{OPT} \\
&\le \mathrm{OPT} \sum_{\ell=1}^{s} \left(\frac{1}{n_\ell} + \frac{1}{n_\ell - 1} + \cdots + \frac{1}{n_{\ell+1} + 1} \right)
\end{aligned}
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$$= \mathrm{OPT} \sum_{i=1}^{k} \frac{1}{i}$$

Technique 4: The Greedy Algorithm

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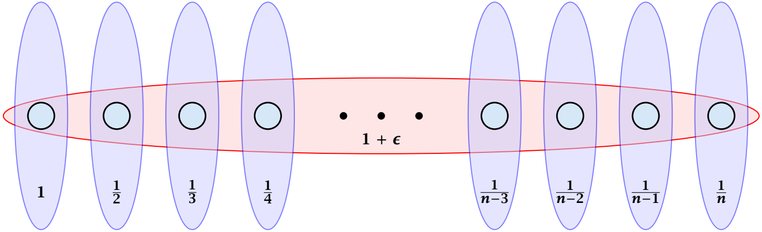
&= \mathrm{OPT}\sum_{i=1}^{k}\frac{1}{i} \\
&= H_n\cdot \mathrm{OPT} \le \mathrm{OPT}(\ln n + 1)\ .
\end{aligned}
$$

Technique 4: The Greedy Algorithm

A tight example:



Technique 5: Randomized Rounding

One round of randomized rounding:
Pick set S_j uniformly at random with probability $1 - x_j$ (for all j).

Version A: Repeat rounds until you have a cover.

Version B: Repeat for s rounds. If you have a cover STOP.
Otherwise, repeat the whole algorithm.

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Probability that $u \in U$ is not covered (in one round):

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\end{aligned}$$

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Probability that $u \in U$ is not covered (after ℓ rounds):

$$\Pr[u \text{ not covered after } \ell \text{ round}] \le \frac{1}{e^{\ell}} \enspace .$$

$\Pr[\exists u \in U$ not covered after ℓ round$]$

$$\Pr[\exists u \in U \text{ not covered after } \ell \text{ round}]$$
$$= \Pr[u_1 \text{ not covered} \vee u_2 \text{ not covered} \vee \ldots \vee u_n \text{ not covered}]$$

$$\begin{aligned}
&\Pr[\exists u \in U \text{ not covered after } \ell \text{ round}] \\
&\qquad = \Pr[u_1 \text{ not covered} \vee u_2 \text{ not covered} \vee \ldots \vee u_n \text{ not covered}] \\
&\qquad \le \sum_i \Pr[u_i \text{ not covered after } \ell \text{ rounds}]
\end{aligned}$$

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$$= \Pr[u_1 \text{ not covered} \vee u_2 \text{ not covered} \vee \ldots \vee u_n \text{ not covered}]$$
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Lemma 16

With high probability $\mathcal{O}(\log n)$ *rounds suffice.*

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With high probability $\mathcal{O}(\log n)$ *rounds suffice.*

With high probability:

For any constant α the number of rounds is at most $\mathcal{O}(\log n)$ with probability at least $1 - n^{-\alpha}$.

Proof: We have

$$\Pr[\#\text{rounds} \geq (\alpha+1)\ln n] \leq ne^{-(\alpha+1)\ln n} = n^{-\alpha} \;.$$

Expected Cost

- Version A.
  Repeat for $s = (\alpha + 1)\ln n$ rounds. If you don't have a cover simply take for each element u the cheapest set that contains u.

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\begin{aligned}
E[\text{cost}] = \Pr[\text{success}] \cdot E[\text{cost} \mid \text{success}] \\
+ \Pr[\text{no success}] \cdot E[\text{cost} \mid \text{no success}]
\end{aligned}
$$

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This means

$$E[\text{cost} \mid \text{success}] = \frac{1}{\Pr[\text{succ.}]}\Big(E[\text{cost}] - \Pr[\text{no success}] \cdot E[\text{cost} \mid \text{no success}]\Big)$$

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This means

$$\begin{aligned} &E[\text{cost} \mid \text{success}] \\ &\quad = \frac{1}{\Pr[\text{succ.}]}\Big(E[\text{cost}] - \Pr[\text{no success}] \cdot E[\text{cost} \mid \text{no success}]\Big) \\ &\quad \le \frac{1}{\Pr[\text{succ.}]} E[\text{cost}] \le \frac{1}{1 - n^{-\alpha}}(\alpha + 1)\ln n \cdot \text{cost}(\text{LP}) \end{aligned}$$

Expected Cost

- Version B.
  Repeat for $s = (\alpha + 1)\ln n$ rounds. If you don't have a cover simply repeat the whole process.

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for $n \ge 2$ and $\alpha \ge 1$.

Randomized rounding gives an $\mathcal{O}(\log n)$ approximation. The running time is polynomial with high probability.

Theorem 17 (without proof)

There is no approximation algorithm for set cover with approximation guarantee better than $\frac{1}{2}\log n$ unless NP has quasi-polynomial time algorithms (algorithms with running time $2^{\mathrm{poly}(\log n)}$).

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Integrality Gap

The integrality gap of the SetCover LP is $\Omega(\log n)$.

- $n = 2^k - 1$
- Elements are all vectors $\vec{x}$ over $GF[2]$ of length k (excluding zero vector).
- Every vector $\vec{y}$ defines a set as follows

$$S_{\vec{y}} := \{\vec{x} \mid \vec{x}^T \vec{y} = 1\}$$

- each set contains 2^{k-1} vectors; each vector is contained in 2^{k-1} sets
- $x_i = \frac{1}{2^{k-1}} = \frac{2}{n+1}$ is fractional solution.

Integrality Gap

Every collection of $p < k$ sets does not cover all elements.

Hence, we get a gap of $\Omega(\log n)$.

Techniques:

- Deterministic Rounding
- Rounding of the Dual
- Primal Dual
- Greedy
- Randomized Rounding
- Local Search
- Rounding Data + Dynamic Programming

Scheduling Jobs on Identical Parallel Machines

Given n jobs, where job $j \in \{1, \ldots, n\}$ has processing time p_j. Schedule the jobs on m identical parallel machines such that the Makespan (finishing time of the last job) is minimized.

$$
\begin{array}{llrcl}
\min & & & L & \\
\text{s.t.} & \forall \text{machines } i & \sum_j p_j \cdot x_{j,i} & \le & L \\
 & \forall \text{jobs } j & \sum_i x_{j,i} \ge 1 & & \\
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Here the variable $x_{j,i}$ is the decision variable that describes whether job j is assigned to machine i.

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Lower Bounds on the Solution

Let for a given schedule C_j denote the finishing time of machine j, and let $C_{\max}$ be the makespan.

Let $C^*_{\max}$ denote the makespan of an optimal solution.

Clearly

$$C^*_{\max} \geq \max_j p_j$$

as the longest job needs to be scheduled somewhere.

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Local Search

A local search algorithm successively makes certain small (cost/profit improving) changes to a solution until it does not find such changes anymore.

It is conceptionally very different from a Greedy algorithm as a feasible solution is always maintained.

Sometimes the running time is difficult to prove.

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Local Search for Scheduling

Local Search Strategy: Take the job that finishes last and try to move it to another machine. If there is such a move that reduces the makespan, perform the switch.

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Local Search Analysis

Let ℓ be the job that finishes last in the produced schedule.

Let S_ℓ be its start time, and let C_ℓ be its completion time.

Note that every machine is busy before time S_ℓ, because otherwise we could move the job ℓ and hence our schedule would not be locally optimal.

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We can split the total processing time into two intervals one from 0 to S_ℓ the other from S_ℓ to C_ℓ.

The interval $[S_\ell, C_\ell]$ is of length $p_\ell \le C^*_{\max}$.

During the first interval $[0, S_\ell]$ all processors are busy, and, hence, the total work performed in this interval is

$$m \cdot S_\ell \le \sum_{j \ne \ell} p_j \enspace .$$

Hence, the length of the schedule is at most

$$p_\ell + \frac{1}{m}\sum_{j \ne \ell} p_j = (1 - \frac{1}{m})p_\ell + \frac{1}{m}\sum_j p_j \le (2 - \frac{1}{m})C^*_{\max}$$

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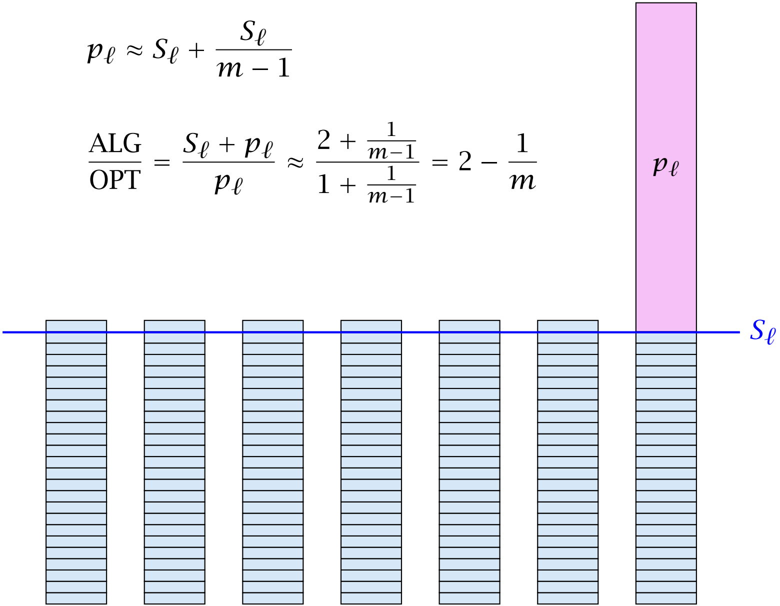
A Tight Example

$$p_\ell \approx S_\ell + \frac{S_\ell}{m-1}$$

$$\frac{\text{ALG}}{\text{OPT}} = \frac{S_\ell + p_\ell}{p_\ell} \approx \frac{2 + \frac{1}{m-1}}{1 + \frac{1}{m-1}} = 2 - \frac{1}{m}$$



A Greedy Strategy

List Scheduling:
Order all processes in a list. When a machine runs empty assign the next yet unprocessed job to it.

Alternatively:
Consider processes in some order. Assign the i-th process to the least loaded machine.

It is easy to see that the result of these greedy strategies fulfill the local optimally condition of our local search algorithm. Hence, these also give 2-approximations.

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A Greedy Strategy

Lemma 18
If we order the list according to non-increasing processing times the approximation guarantee of the list scheduling strategy improves to $4/3$.

Proof:

- Let $p_1 \geq \dots \geq p_n$ denote the processing times of a set of jobs that form a counter-example.
- Wlog. the last job to finish is n (otw. deleting this job gives another counter-example with fewer jobs).
- If $p_n \leq C^*_{\max}/3$ the previous analysis gives us a schedule length of at most

$$C^*_{\max} + p_n \leq \frac{4}{3} C^*_{\max} \enspace .$$

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  Hence, $p_n > C^*_{\max}/3$.
- This means that all jobs must have a processing time $> C^*_{\max}/3$.
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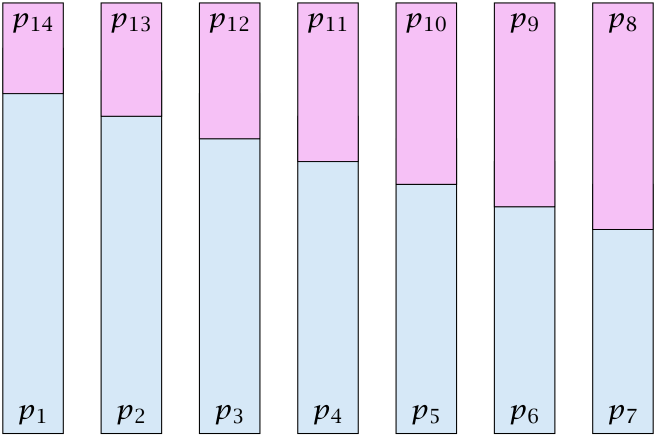
When in an optimal solution a machine can have at most 2 jobs the optimal solution looks as follows.



- We can assume that one machine schedules p_1 and p_n (the largest and smallest job).
- If not assume wlog. that p_1 is scheduled on machine A and p_n on machine B.
- Let p_A and p_B be the other job scheduled on A and B, respectively.
- $p_1 + p_n \le p_1 + p_A$ and $p_A + p_B \le p_1 + p_A$, hence scheduling p_1 and p_n on one machine and p_A and p_B on the other, cannot increase the Makespan.
- Repeat the above argument for the remaining machines.

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- $2m+1$ jobs

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- 2 jobs with length $2m, 2m-1, 2m-2, \dots, m+1$ ($2m-2$ jobs in total)

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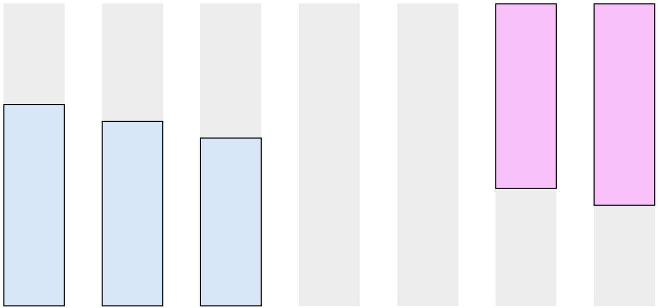

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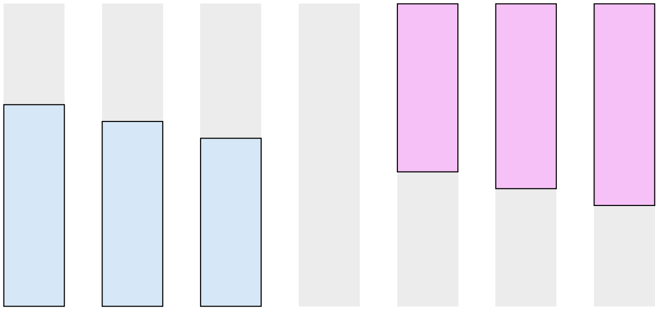

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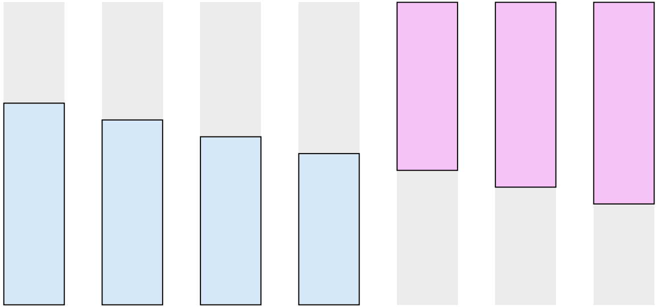

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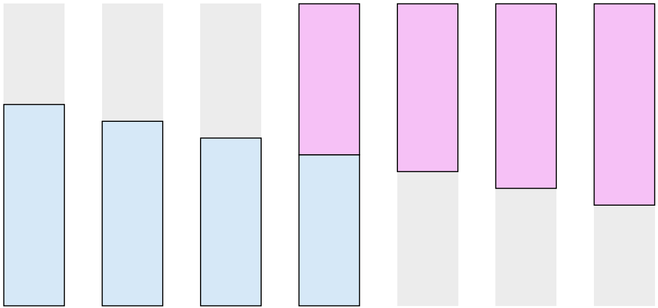

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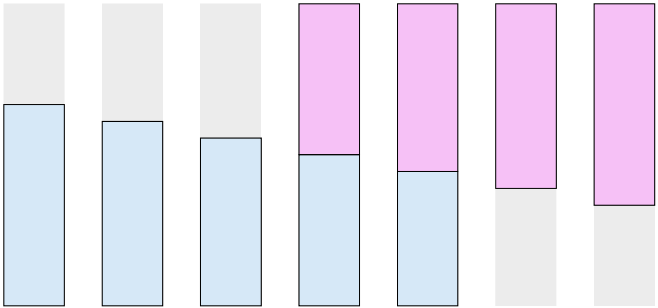

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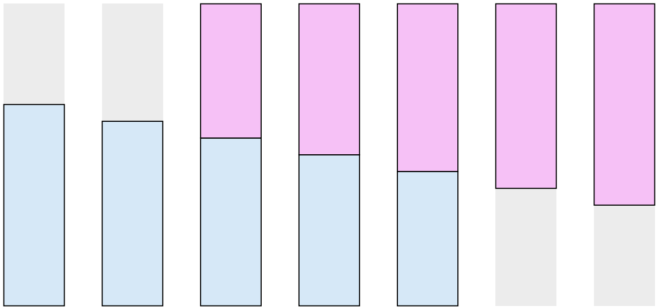

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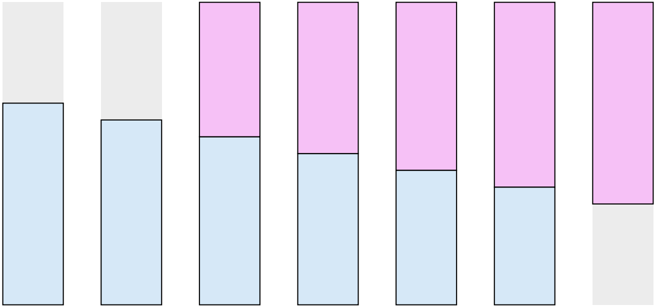

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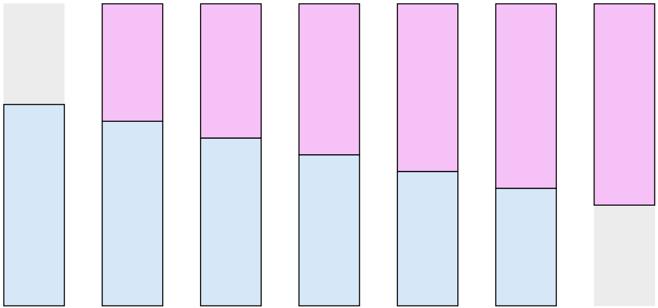

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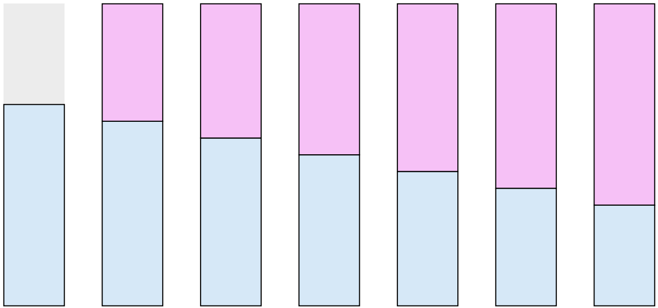

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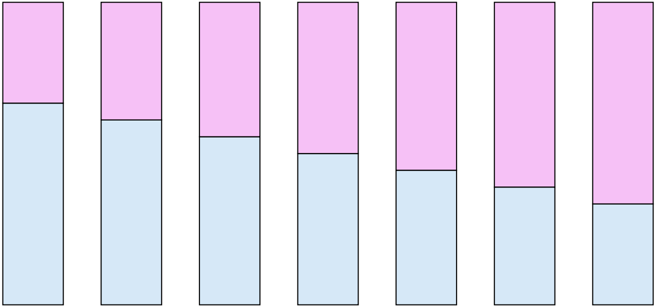

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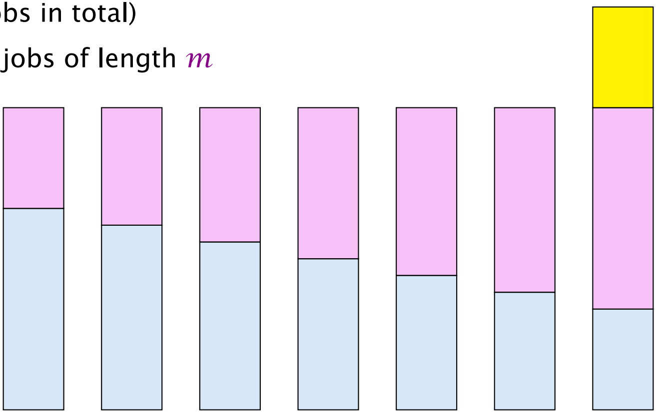

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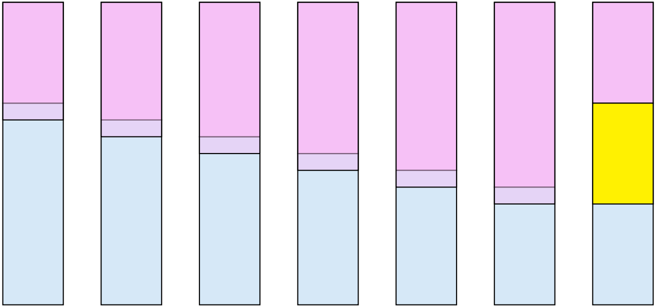

Traveling Salesman

Given a set of cities ($\{1,\ldots,n\}$) and a symmetric matrix $C = (c_{ij})$, $c_{ij} \geq 0$ that specifies for every pair $(i,j) \in [n] \times [n]$ the cost for travelling from city i to city j. Find a permutation π of the cities such that the round-trip cost

$$c_{\pi(1)\pi(n)} + \sum_{i=1}^{n-1} c_{\pi(i)\pi(i+1)}$$

is minimized.

Traveling Salesman

Theorem 19

There does not exist an $O(2^n)$-approximation algorithm for TSP.

Hamiltonian Cycle:
For a given undirected graph $G = (V, E)$ decide whether there exists a simple cycle that contains all nodes in G.

- Given an instance to HAMPATH we create an instance for TSP.
- If $(i,j) \notin E$ then set c_{ij} to $n2^n$ otw. set c_{ij} to 1. This instance has polynomial size.
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$$c_{ij} \le c_{ij} + c_{jk} \quad .$$

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TSP: Lower Bound I

Lemma 20

The cost $\mathrm{OPT}_{TSP}(G)$ *of an optimum traveling salesman tour is at least as large as the weight* $\mathrm{OPT}_{MST}(G)$ *of a minimum spanning tree in* G.

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- Take the node v closest to S. Add it S and expand the existing tour on S to include v.
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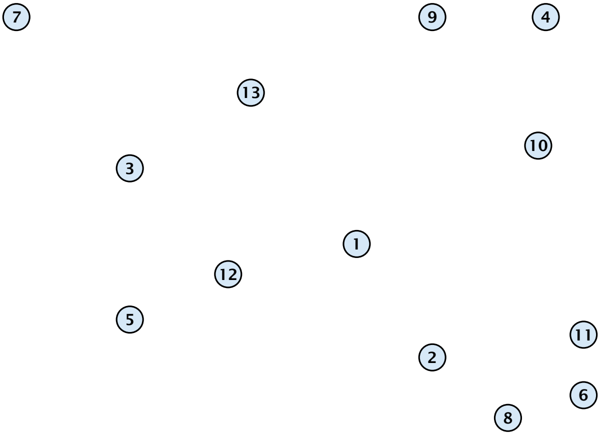

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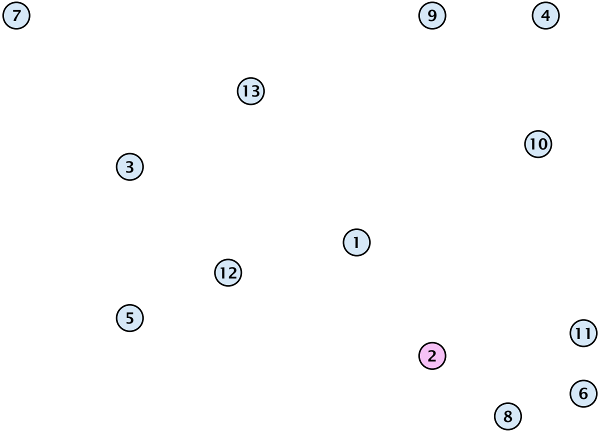

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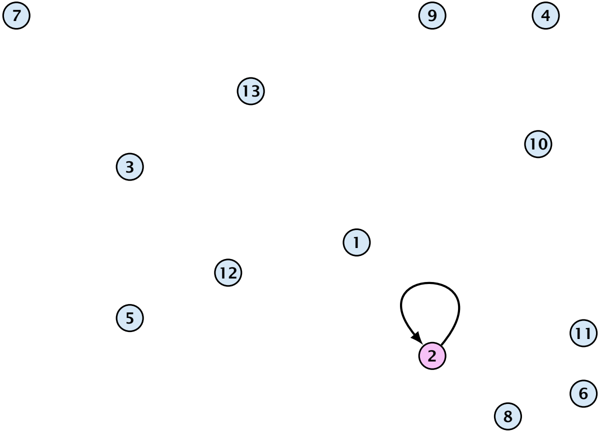

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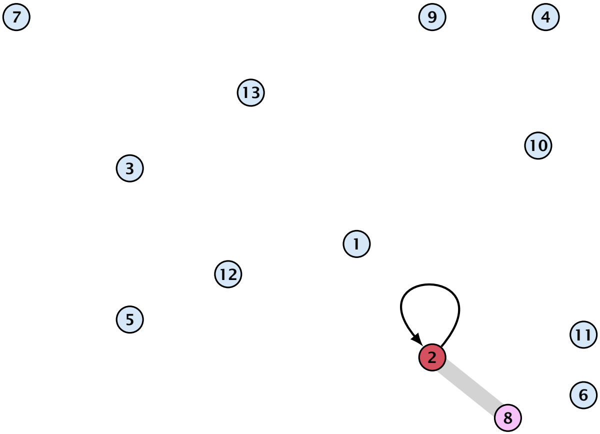

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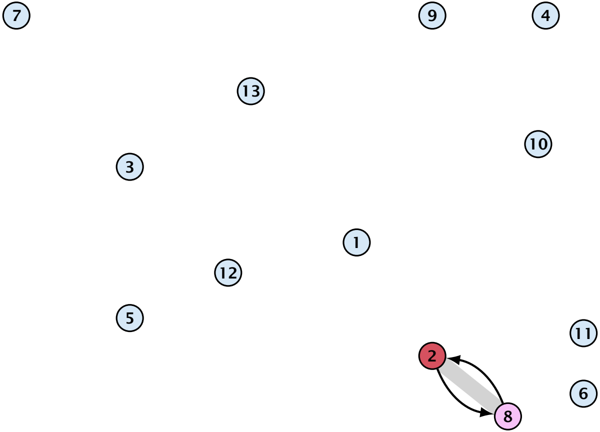

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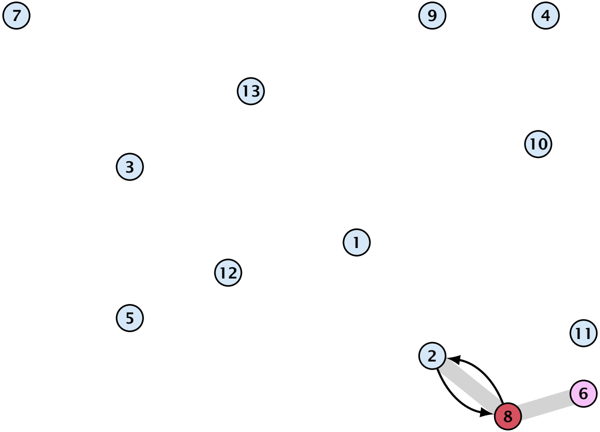

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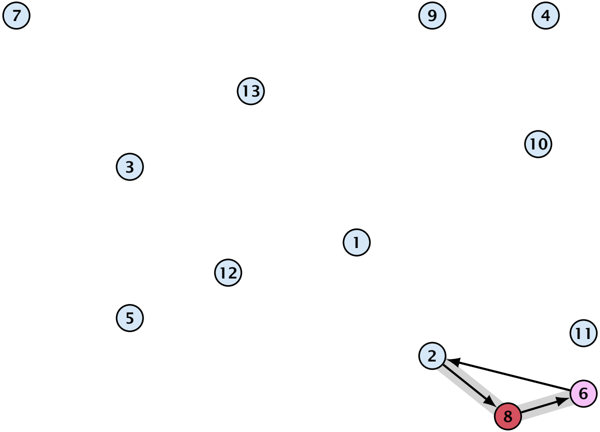

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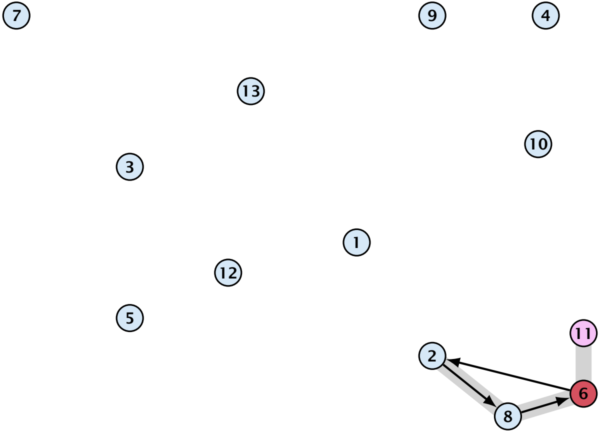

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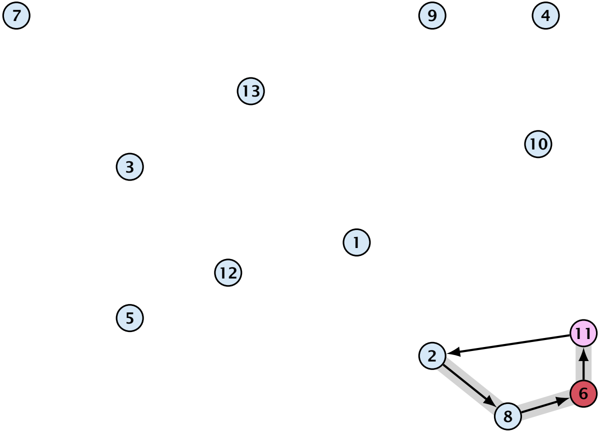

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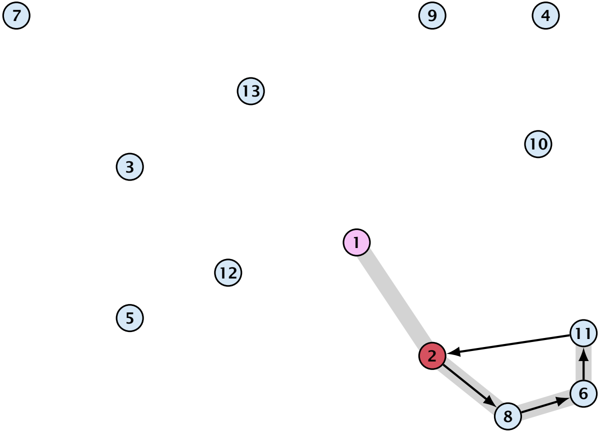

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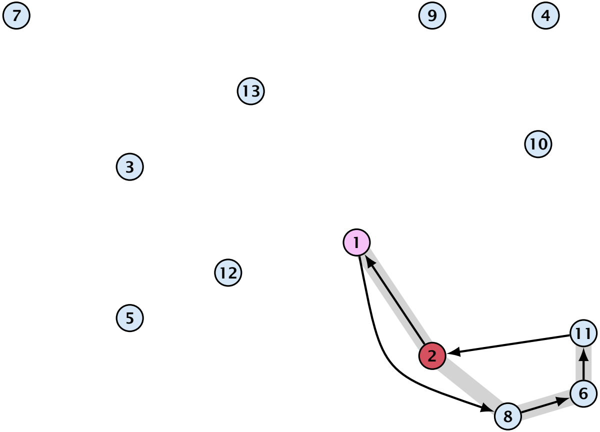

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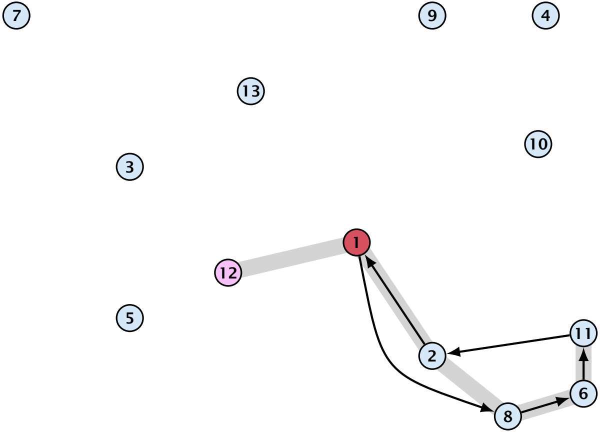

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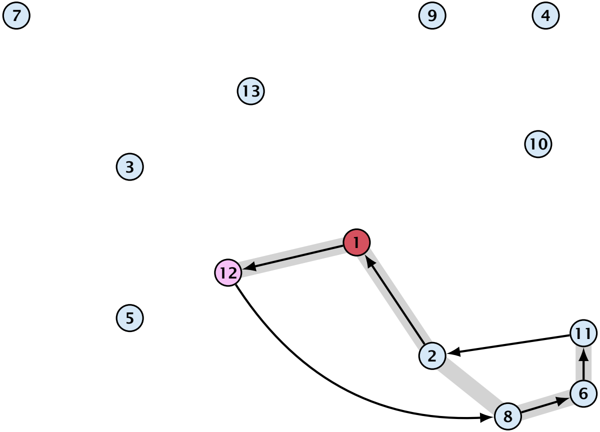

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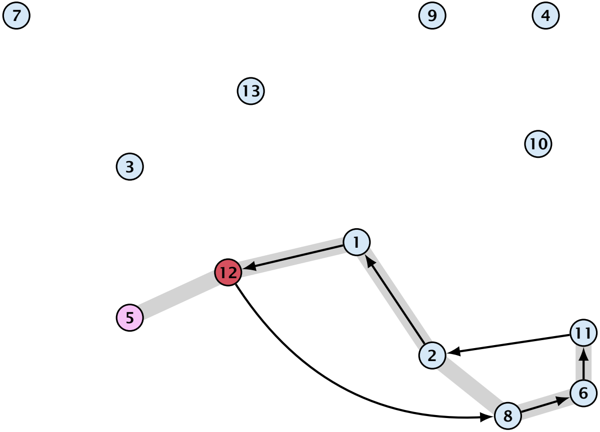

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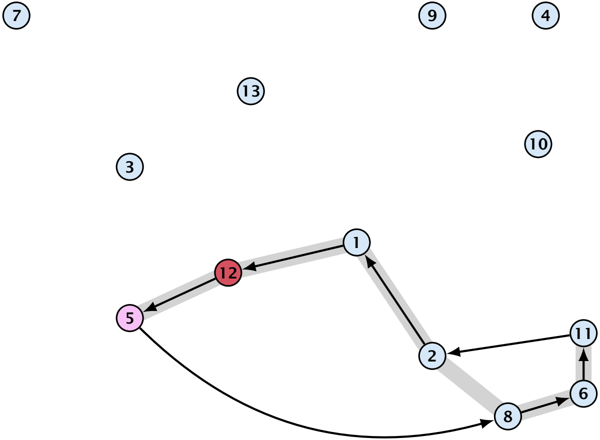

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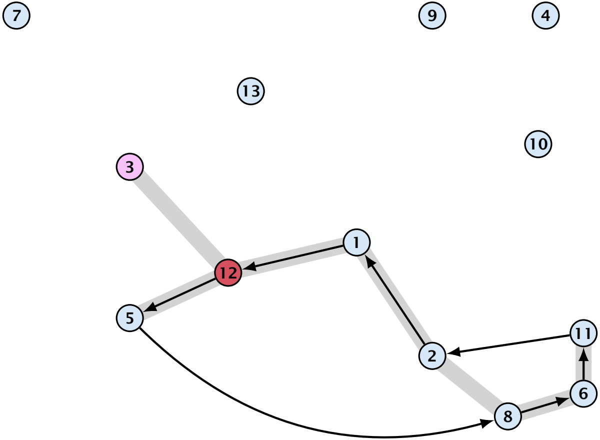

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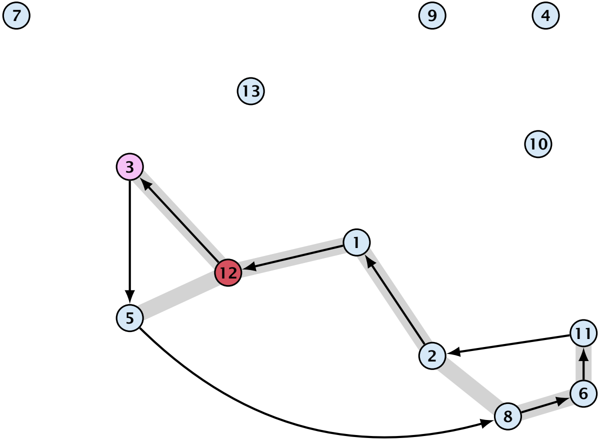

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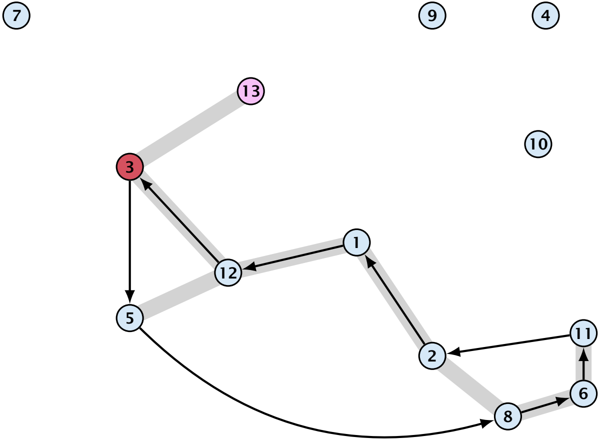

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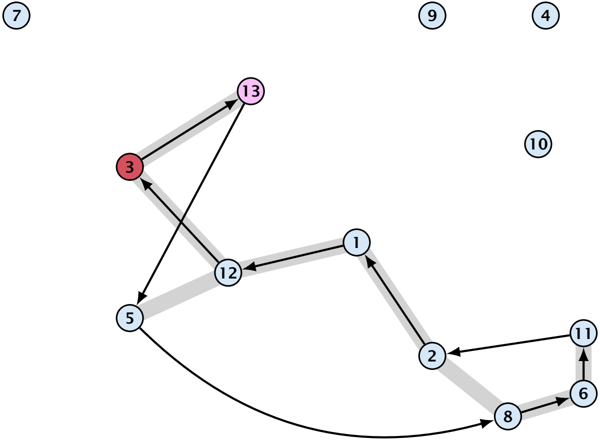

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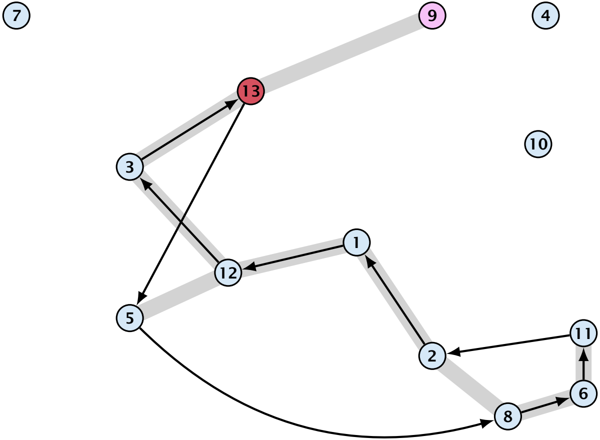

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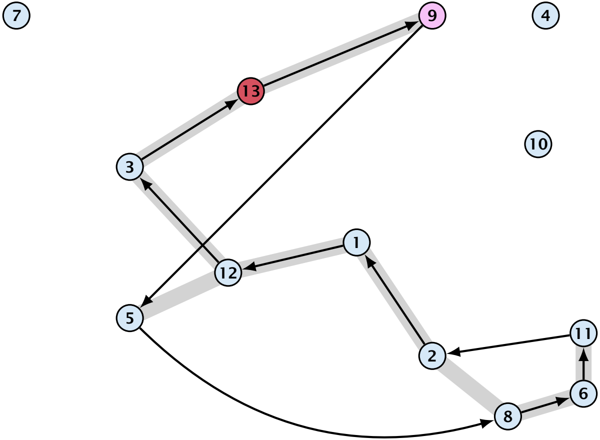

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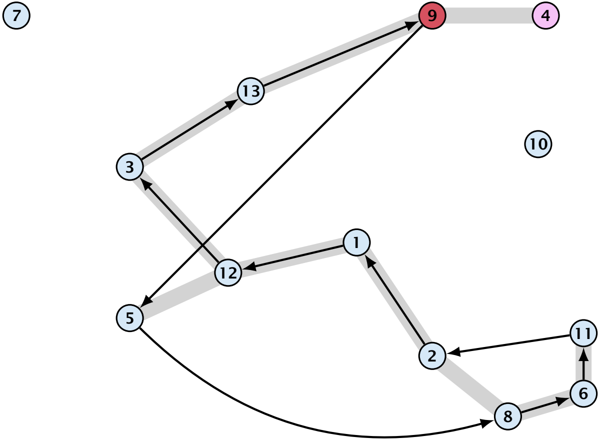

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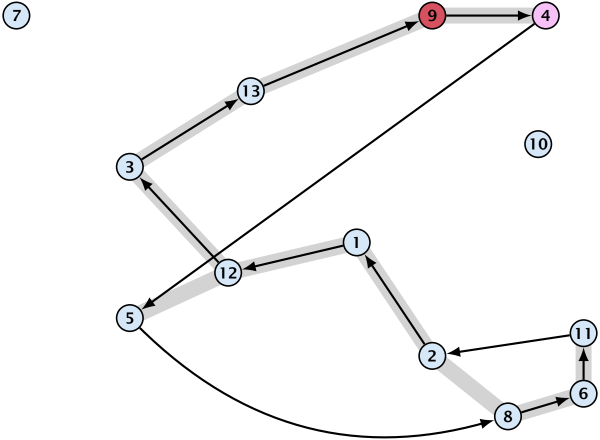

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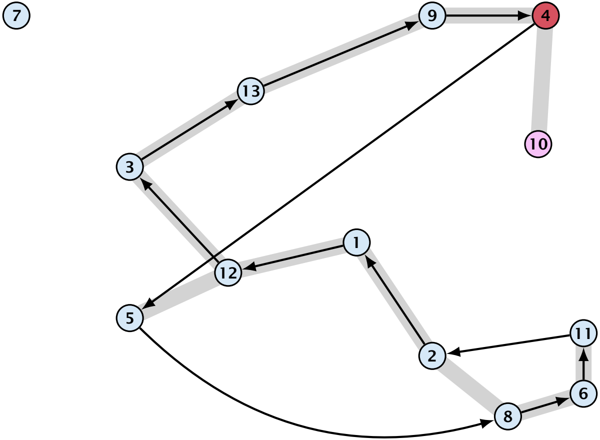

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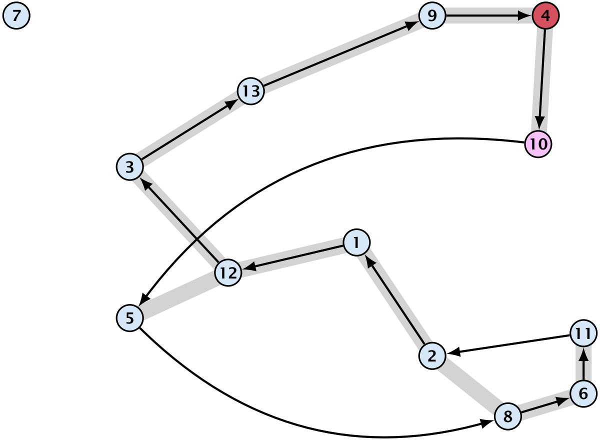

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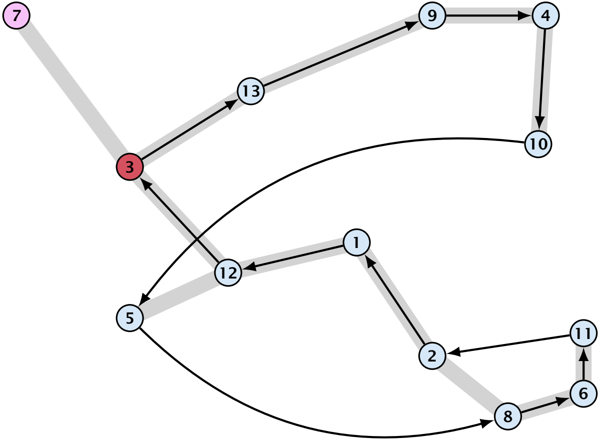

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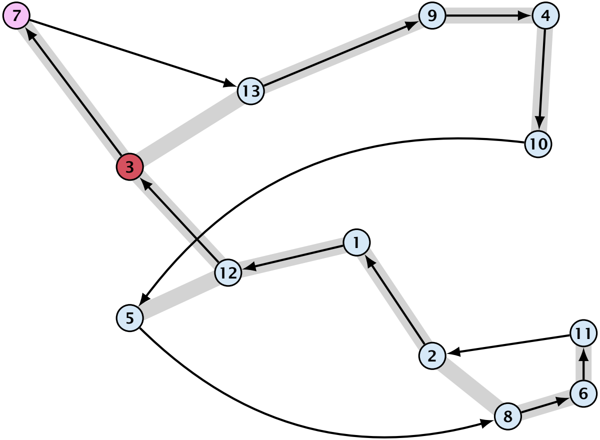

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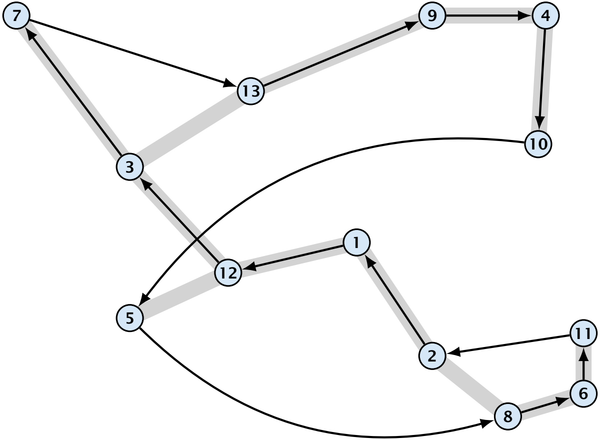

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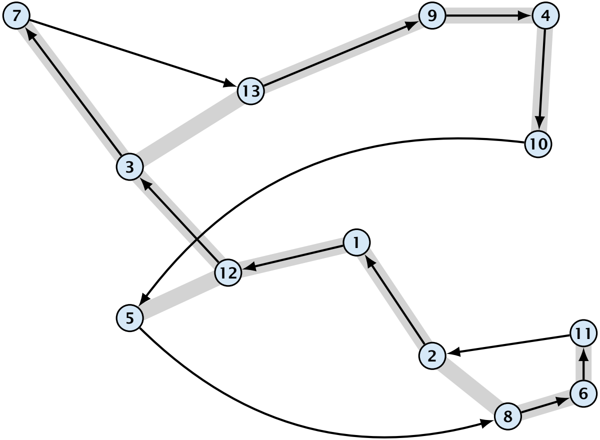

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The Greedy algorithm is a 2-approximation algorithm.

Let S_i be the set at the start of the i-th iteration, and let v_i denote the node added during the iteration.

Further let $s_i \in S_i$ be the node closest to $v_i \in S_i$.

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We replace the edge (s_i, r_i) in the tour by the two edges (s_i, v_i) and (v_i, r_i).

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The edges (s_i, v_i) considered during the Greedy algorithm are exactly the edges considered during PRIMs MST algorithm.

Hence,

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TSP: A different approach

Suppose that we are given an Eulerian graph $G' = (V, E', c')$ of $G = (V, E, c)$ such that for any edge $(i,j) \in E'$ $c'(i,j) \geq c(i,j)$.

Then we can find a TSP-tour of cost at most

$$\sum_{e \in E'} c'(e)$$

- Find an Euler tour of G'.
- Fix a permutation of the cities (i.e., a TSP-tour) by traversing the Euler tour and only note the first occurrence of a city.
- The cost of this TSP tour is at most the cost of the Euler tour because of triangle inequality.

This technique is known as short cutting the Euler tour.

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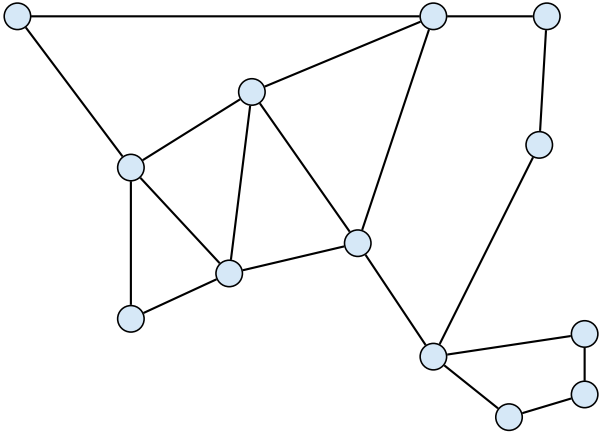

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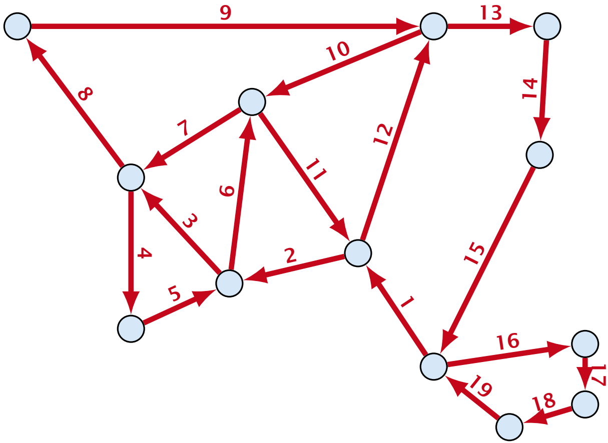

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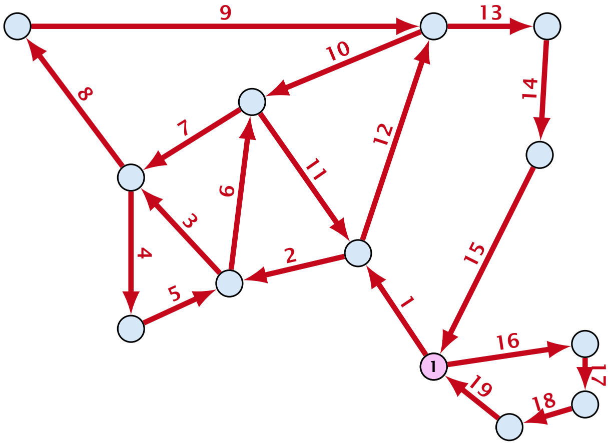

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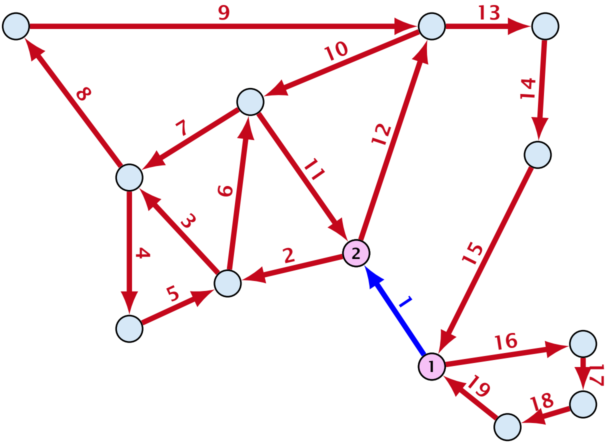

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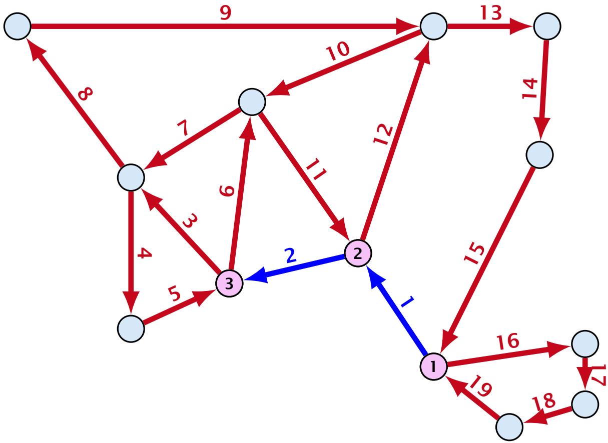

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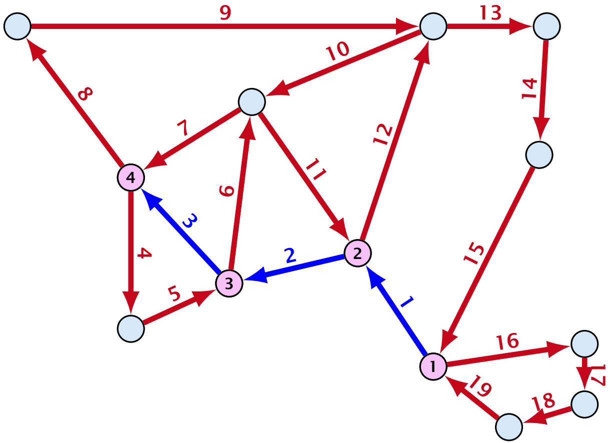

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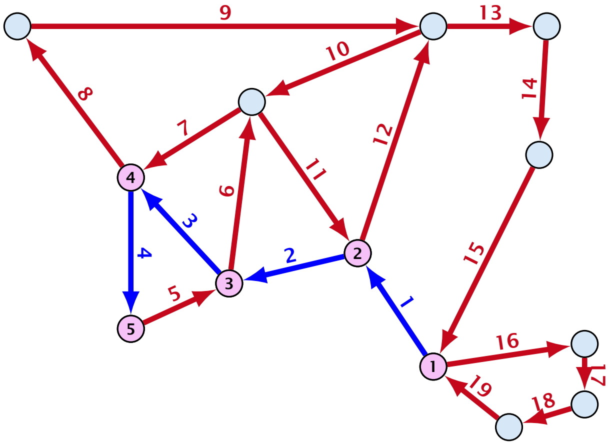

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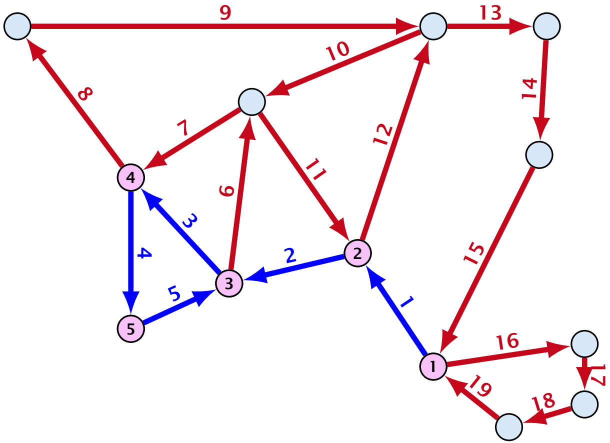

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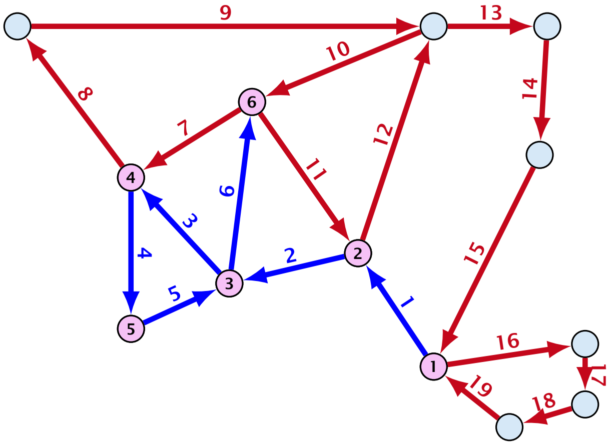

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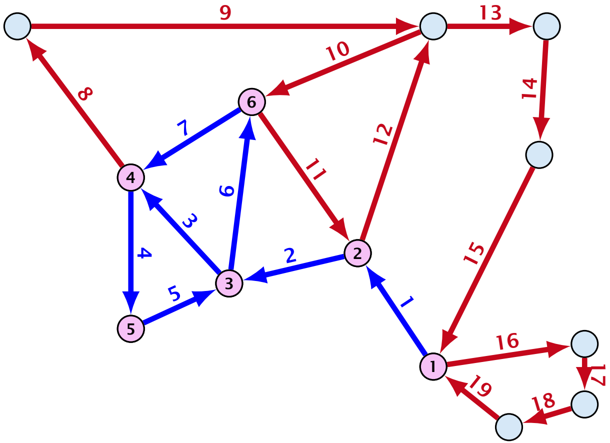

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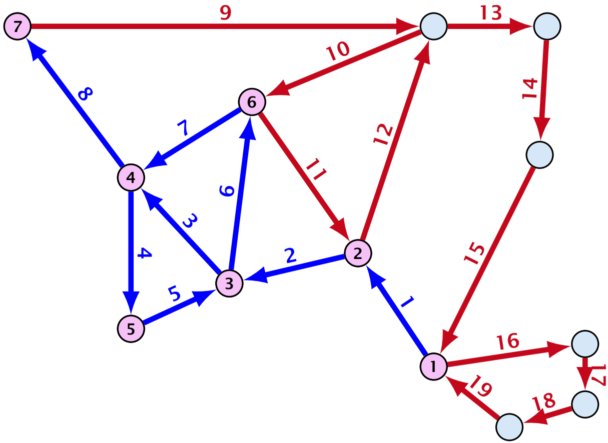

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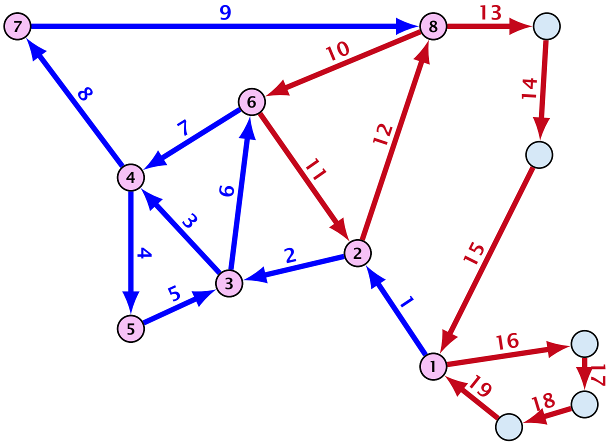

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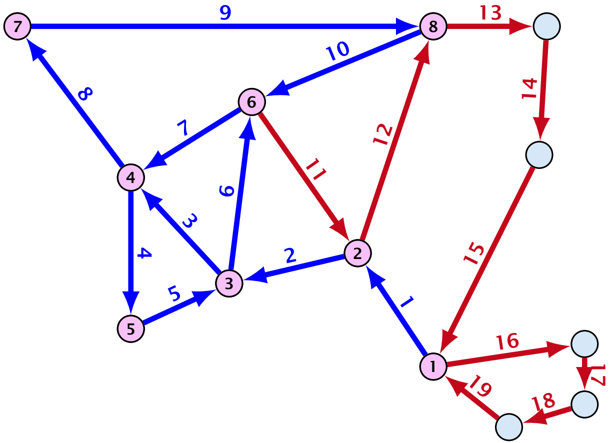

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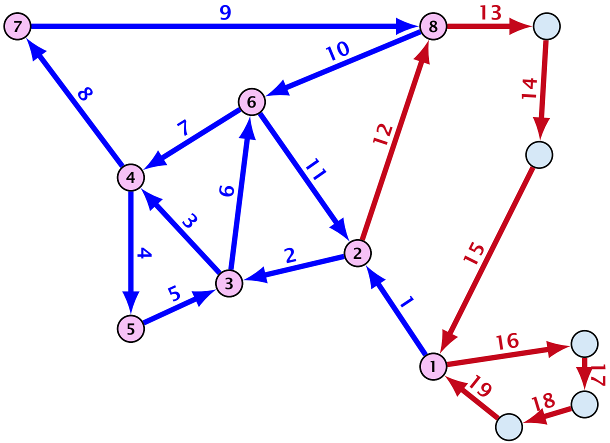

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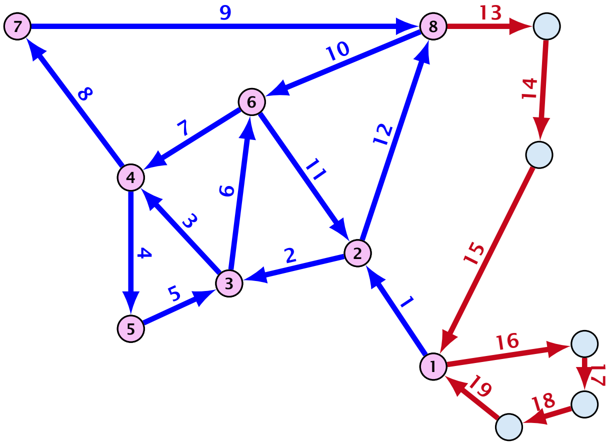

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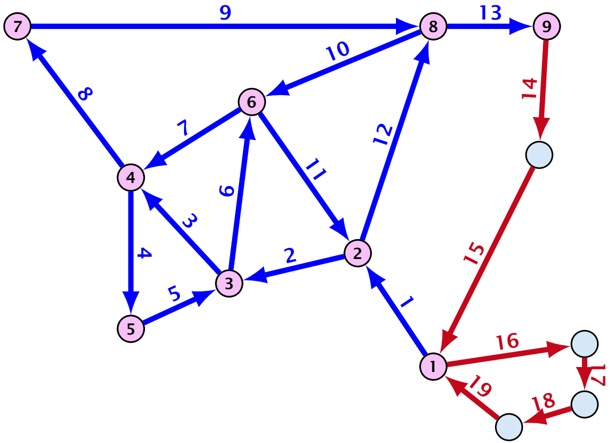

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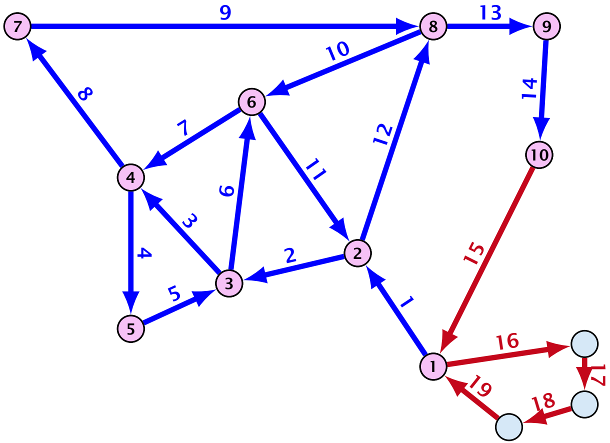

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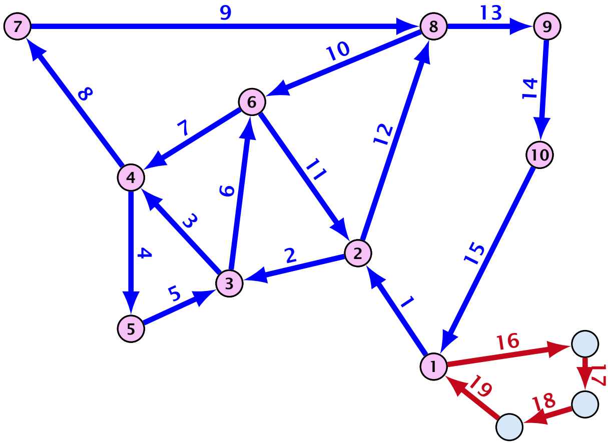

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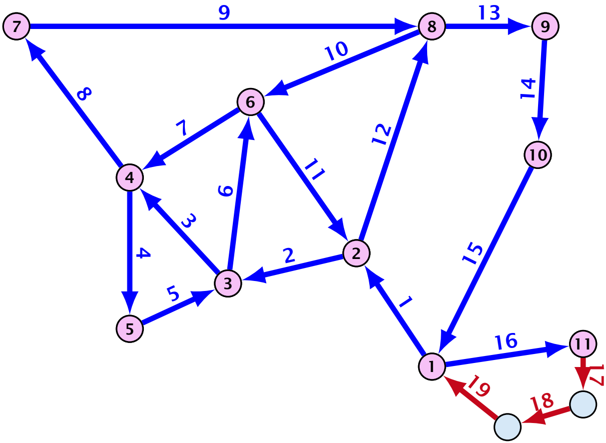

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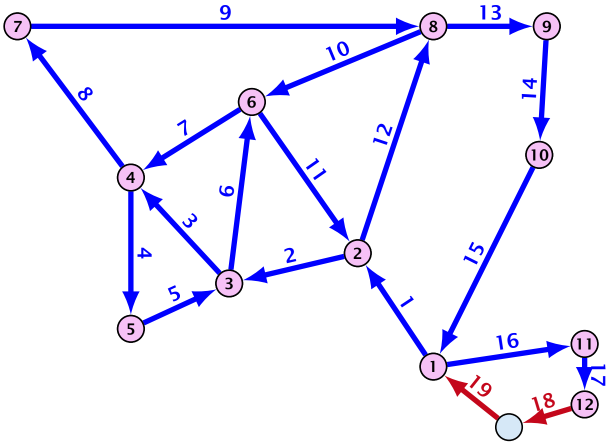

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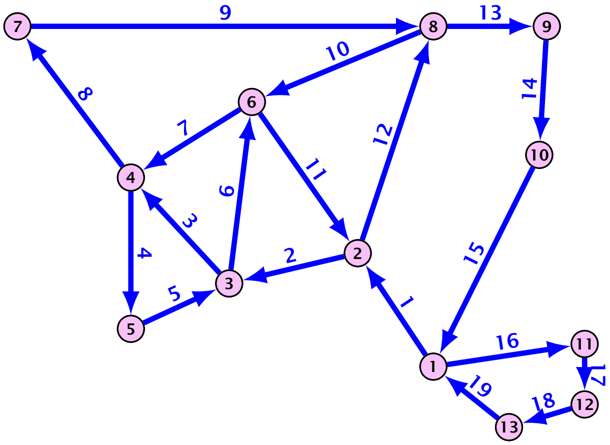

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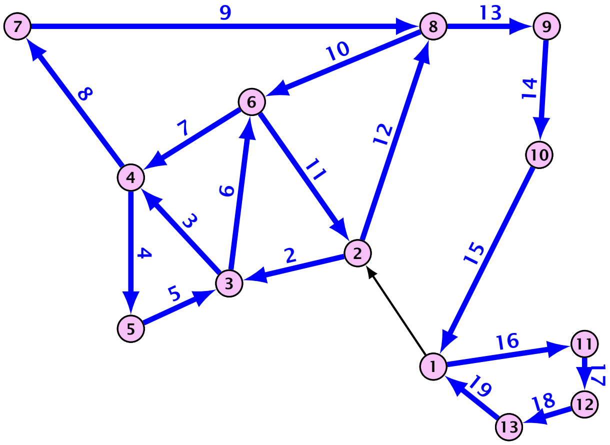

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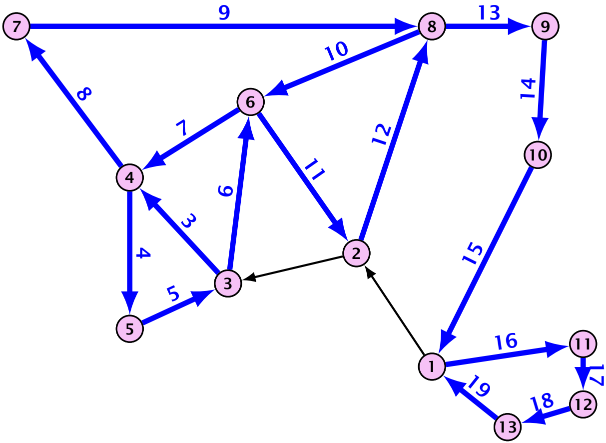

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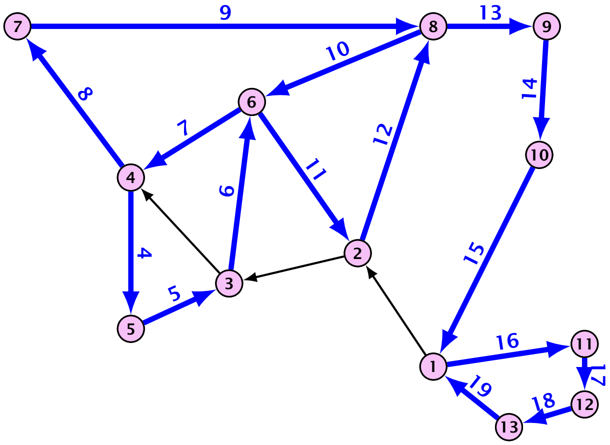

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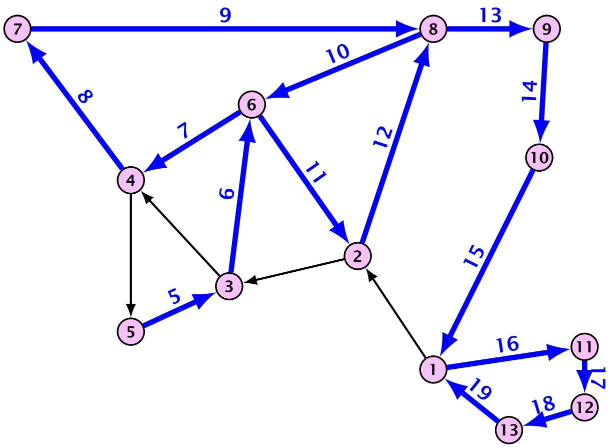

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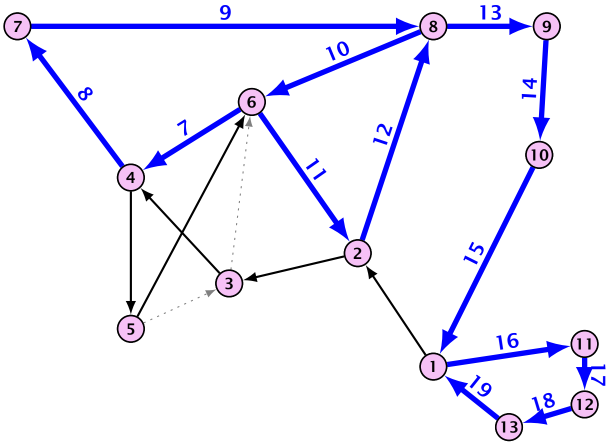

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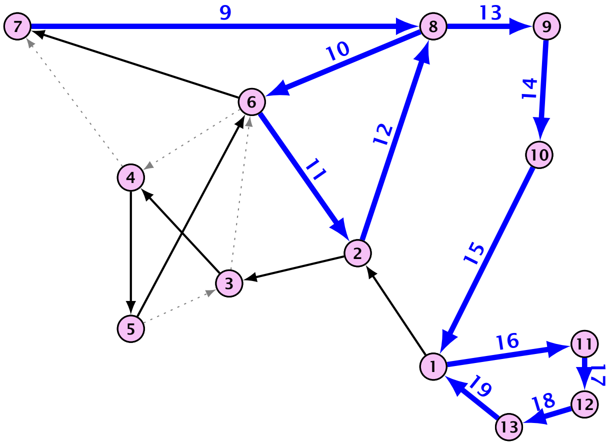

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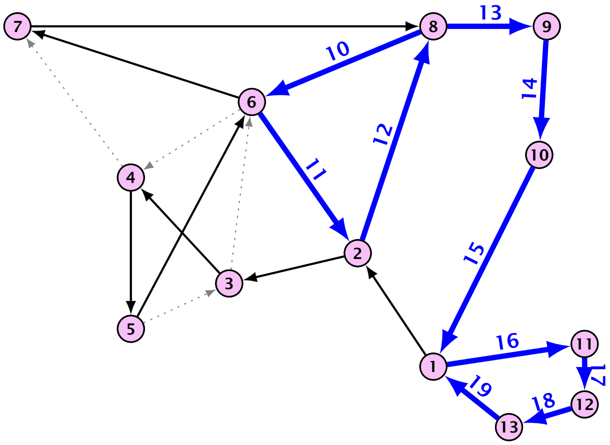

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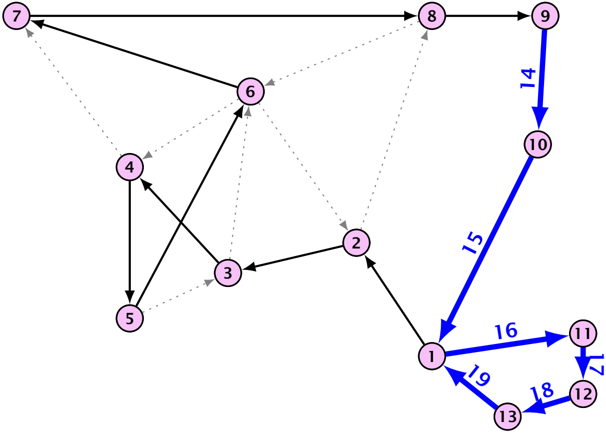

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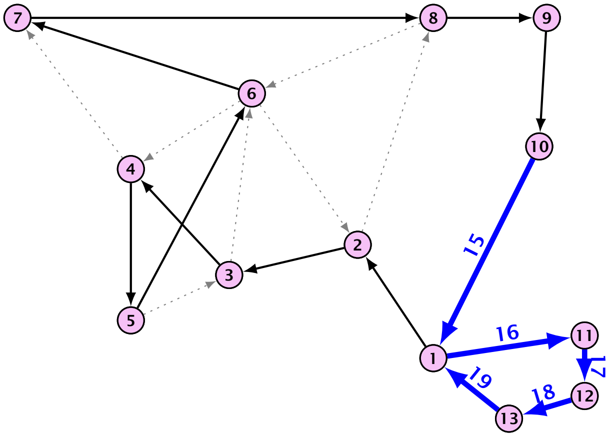

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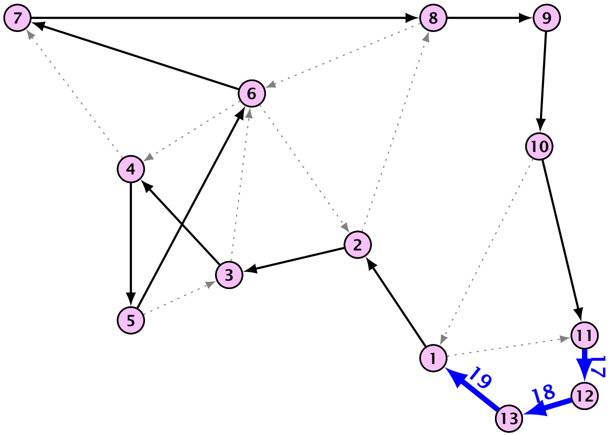

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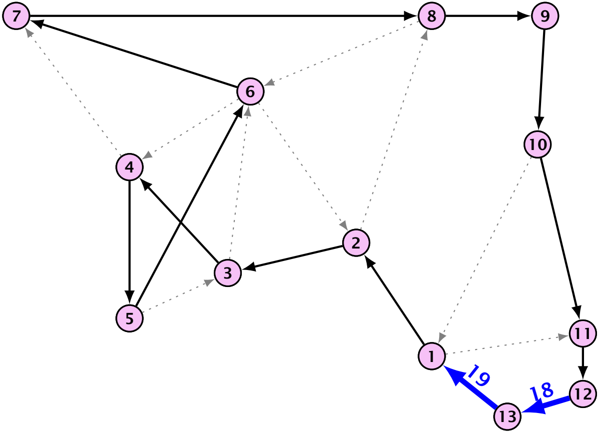

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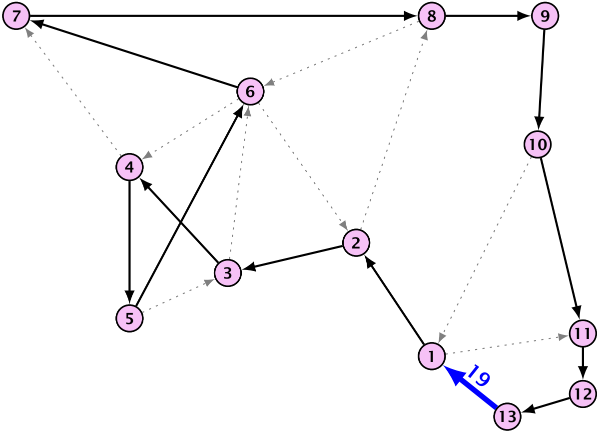

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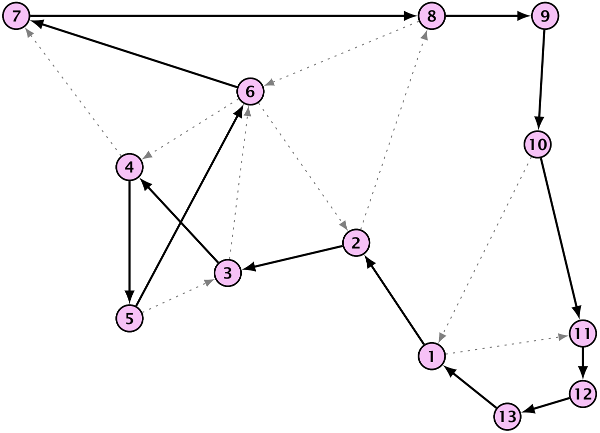

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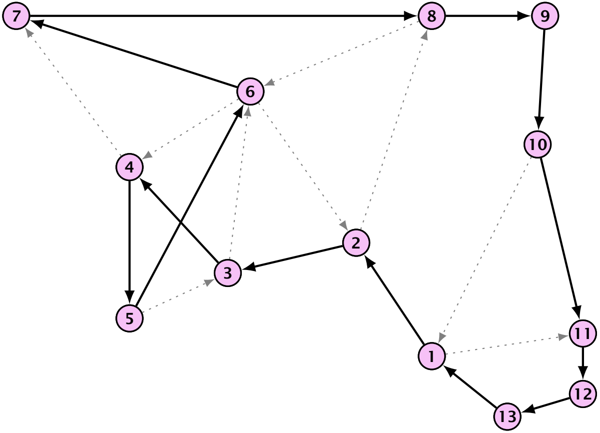

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Consider the following graph:

- Compute an MST of G.
- Duplicate all edges.

This graph is Eulerian, and the total cost of all edges is at most $2 \cdot \text{OPT}_{\text{MST}}(G)$.

Hence, short-cutting gives a tour of cost no more than $2 \cdot \text{OPT}_{\text{MST}}(G)$ which means we have a 2-approximation.

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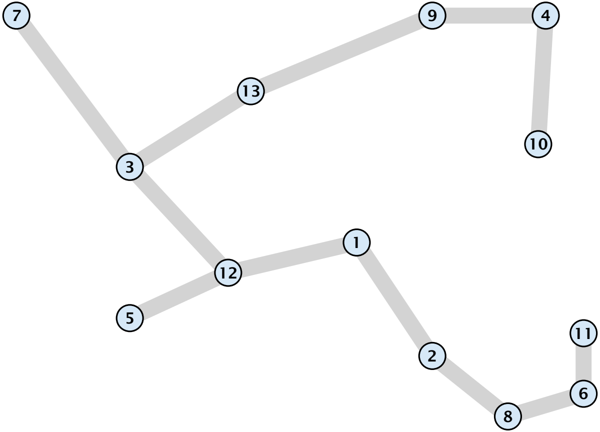

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Adding this matching to the MST gives an Eulerian graph with edge weight at most

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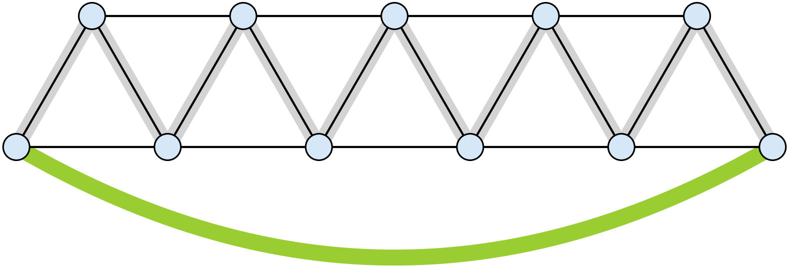
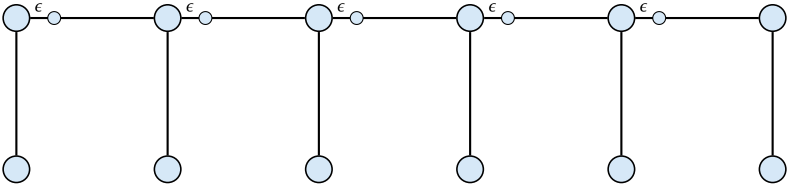
Christofides. Tight Example



- optimal tour: n edges.
- MST: $n-1$ edges.
- weight of matching $(n+1)/2-1$
- MST+matching $\approx {}^3/_2 \cdot n$

Tree shortcutting. Tight Example



- edges have Euclidean distance.

17 Rounding Data + Dynamic Programming

Knapsack:
Given a set of items $\{1, \dots, n\}$, where the i-th item has weight $w_i \in \mathbb{N}$ and profit $p_i \in \mathbb{N}$, and given a threshold W. Find a subset $I \subseteq \{1, \dots, n\}$ of items of total weight at most W such that the profit is maximized (we can assume each $w_i \le W$).

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17 Rounding Data + Dynamic Programming

Algorithm 1 Knapsack

$A(1) \leftarrow [(0,0),(p_1,w_1)]$
for $j \leftarrow 2$ to n **do**
    $A(j) \leftarrow A(j-1)$
    for each $(p,w) \in A(j-1)$ **do**
        if $w + w_j \le W$ **then**
            add $(p + p_j, w + w_j)$ to $A(j)$
        remove dominated pairs from $A(j)$
return $\max_{(p,w)\in A(n)} p$

The running time is $\mathcal{O}(n \cdot \min\{W, P\})$, where $P = \sum_i p_i$ is the total profit of all items. This is only pseudo-polynomial.

17 Rounding Data + Dynamic Programming

Definition 22
An algorithm is said to have pseudo-polynomial running time if the running time is polynomial when the numerical part of the input is encoded in unary.

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Scheduling Revisited

The previous analysis of the scheduling algorithm gave a makespan of

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Together with the obervation that if each $p_i \geq \frac{1}{3}C^*_{\max}$ then LPT is optimal this gave a $4/3$-approximation.

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Idea:

1. Find the optimum Makespan for the long jobs by brute force.
2. Then use the list scheduling algorithm for the short jobs, always assigning the next job to the least loaded machine.

We still have the inequality

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If ℓ is a short job its length is at most

$$p_\ell \le \sum_j p_j/(mk)$$

which is at most $C^*_{\max}/k$.

Hence we get a schedule of length at most

$$\left(1 + \frac{1}{k}\right) C^*_{\max}$$

There are at most km long jobs. Hence, the number of possibilities of scheduling these jobs on m machines is at most m^{km}, which is constant if m is constant. Hence, it is easy to implement the algorithm in polynomial time.

Theorem 23

The above algorithm gives a polynomial time approximation scheme (PTAS) for the problem of scheduling n jobs on m identical machines if m is constant.

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How to get rid of the requirement that m is constant?

We first design an algorithm that works as follows:
On input of T it either finds a schedule of length $(1 + \frac{1}{k})T$ or certifies that no schedule of length at most T exists (assume $T \geq \frac{1}{m}\sum_j p_j$).

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- We round all long jobs down to multiples of T/k^2.
- For these rounded sizes we first find an optimal schedule.
- If this schedule does not have length at most T we conclude that also the original sizes don't allow such a schedule.
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After the first phase the rounded sizes of the long jobs assigned to a machine add up to at most T.

There can be at most k (long) jobs assigned to a machine as otw. their rounded sizes would add up to more than T (note that the rounded size of a long job is at least T/k).

Since, jobs had been rounded to multiples of T/k^2 going from rounded sizes to original sizes gives that the Makespan is at most

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During the second phase there always must exist a machine with load at most T, since T is larger than the average load.

Assigning the current (short) job to such a machine gives that the new load is at most

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Running Time for scheduling large jobs: There should not be a job with rounded size more than T as otw. the problem becomes trivial.

Hence, any large job has rounded size of $\frac{i}{k^2}T$ for $i \in \{k, \dots, k^2\}$. Therefore the number of different inputs is at most n^{k^2} (described by a vector of length k^2 where, the i-th entry describes the number of jobs of size $\frac{i}{k^2}T$). This is polynomial.

The schedule/configuration of a particular machine x can be described by a vector of length k^2 where the i-th entry describes the number of jobs of rounded size $\frac{i}{k^2}T$ assigned to x. There are only $(k+1)^{k^2}$ different vectors.

This means there are a constant number of different machine configurations.

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Let $\mathrm{OPT}(n_1, \dots, n_{k^2})$ be the number of machines that are required to schedule input vector $(n_1, \dots, n_{k^2})$ with Makespan at most T.

If $\mathrm{OPT}(n_1, \dots, n_{k^2}) \le m$ we can schedule the input.

We have

$$\mathrm{OPT}(n_1, \dots, n_{k^2}) = \begin{cases} 0 & (n_1, \dots, n_{k^2}) = 0 \\ 1 + \min\limits_{(s_1, \dots, s_{k^2}) \in C} \mathrm{OPT}(n_1 - s_1, \dots, n_{k^2} - s_{k^2}) & (n_1, \dots, n_{k^2}) \ngeq 0 \\ \infty & \text{otw.} \end{cases}$$

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Can we do better?

Scheduling on identical machines with the goal of minimizing Makespan is a strongly NP-complete problem.

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There is no FPTAS for problems that are strongly NP-hard.

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- Suppose we have an instance with polynomially bounded processing times $p_i \leq q(n)$
- We set $k := \lceil 2nq(n) \rceil \geq 2\,\mathrm{OPT}$
- Then

$$\mathrm{ALG} \leq \left(1 + \frac{1}{k}\right)\mathrm{OPT} \leq \mathrm{OPT} + \frac{1}{2}$$

- But this means that the algorithm computes the optimal solution as the optimum is integral.
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More General

Let $\mathrm{OPT}(n_1, \ldots, n_A)$ be the number of machines that are required to schedule input vector $(n_1, \ldots, n_A)$ with Makespan at most T (A: number of different sizes).

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where C is the set of all configurations.

$|C| \leq (B+1)^A$, where B is the number of jobs that possibly can fit on the same machine.

The running time is then $O((B+1)^A n^A)$ because the dynamic programming table has just n^A entries.

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Bin Packing

Given n items with sizes $s_1, \dots, s_n$ where

$$1 > s_1 \geq \cdots \geq s_n > 0 \ .$$

Pack items into a minimum number of bins where each bin can hold items of total size at most 1.

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Proof

- In the partition problem we are given positive integers $b_1, \ldots, b_n$ with $B = \sum_i b_i$ even. Can we partition the integers into two sets S and T s.t.

$$\sum_{i \in S} b_i = \sum_{i \in T} b_i \quad ?$$

- We can solve this problem by setting $s_i := 2b_i/B$ and asking whether we can pack the resulting items into 2 bins or not.
- A ρ-approximation algorithm with $\rho < 3/2$ cannot output 3 or more bins when 2 are optimal.
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Definition 26

An asymptotic polynomial-time approximation scheme (APTAS) is a family of algorithms $\{A_\epsilon\}$ along with a constant c such that A_ϵ returns a solution of value at most $(1+\epsilon)\mathrm{OPT}+c$ for minimization problems.

Note that for Set Cover or for Knapsack it makes no sense to differentiate between the notion of a PTAS or an APTAS because of scaling.

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Bin Packing

Again we can differentiate between small and large items.

Lemma 27

Any packing of items into ℓ bins can be extended with items of size at most γ s.t. we use only $\max\{\ell, \frac{1}{1-\gamma}\mathrm{SIZE}(I) + 1\}$ bins, where $\mathrm{SIZE}(I) = \sum_i s_i$ is the sum of all item sizes.

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- If after Greedy we use more than ℓ bins, all bins (apart from the last) must be full to at least $1 - \gamma$.
- Hence, $r(1 - \gamma) \leq \text{SIZE}(I)$ where r is the number of nearly-full bins.
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Choose $\gamma = \epsilon/2$. Then we either use ℓ bins or at most

$$\frac{1}{1-\epsilon/2} \cdot \mathrm{OPT} + 1 \le (1+\epsilon) \cdot \mathrm{OPT} + 1$$

bins.

It remains to find an algorithm for the large items.

Bin Packing

Linear Grouping:

Generate an instance I' (for large items) as follows.

- Order large items according to size.
- Let the first k items belong to group 1; the following k items belong to group 2; etc.
- Delete items in the first group;
- Round items in the remaining groups to the size of the largest item in the group.

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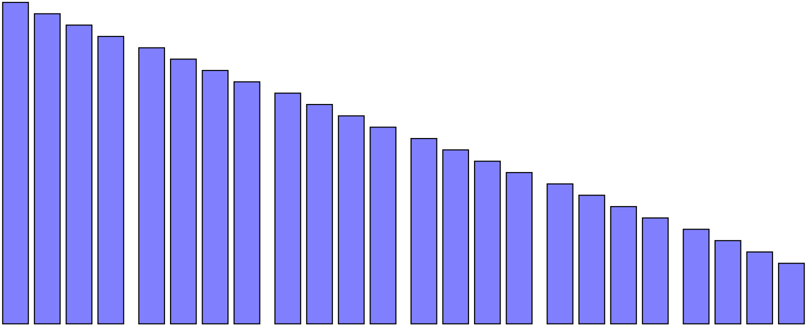

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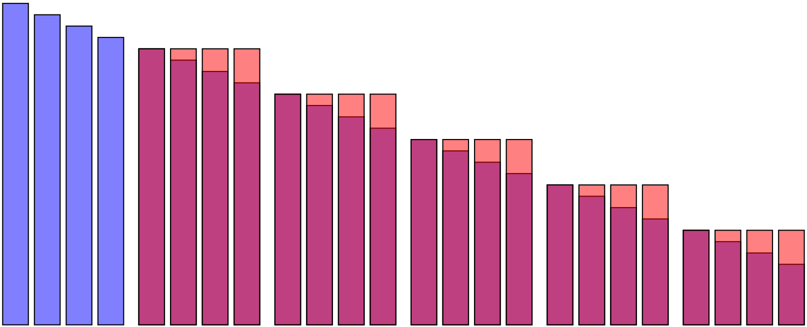

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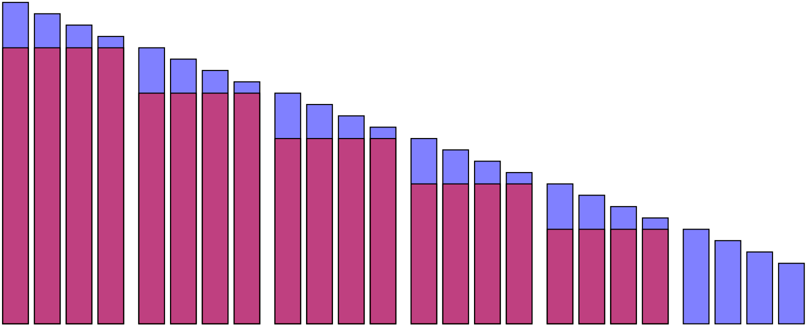

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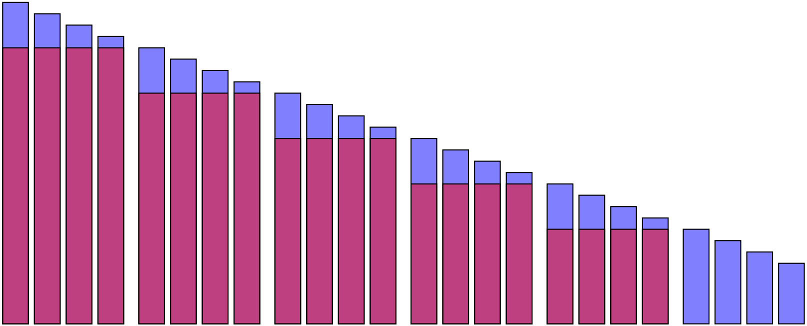

Lemma 28

$\mathrm{OPT}(I') \leq \mathrm{OPT}(I) \leq \mathrm{OPT}(I') + k$

Proof 1:

- Any bin packing for I gives a bin packing for I' as follows.
- Pack the items of group 2, where in the packing for I the items for group 1 have been packed;
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Assume that our instance does not contain pieces smaller than $\epsilon/2$. Then $\text{SIZE}(I) \geq \epsilon n/2$.

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Hence, after grouping we have a constant number of piece sizes ($4/\epsilon^2$) and at most a constant number ($2/\epsilon$) can fit into any bin.

We can find an optimal packing for such instances by the previous Dynamic Programming approach.

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$$\text{OPT}(I') + k \leq \text{OPT}(I) + \epsilon\text{SIZE}(I) \leq (1+\epsilon)\text{OPT}(I)$$

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Change of Notation:

- Group pieces of identical size.
- Let s_1 denote the largest size, and let b_1 denote the number of pieces of size s_1.
- s_2 is second largest size and b_2 number of pieces of size s_2;
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We call a vector that fulfills the above constraint a configuration.

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Let $T_1, \dots, T_N$ be the sequence of all possible configurations (a configuration T_j has T_{ji} pieces of size s_i).

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\min & & \sum_{j=1}^{N} x_j & & \\
\text{s.t.} & \forall i \in \{1 \dots m\} & \sum_{j=1}^{N} T_{ji} x_j & \geq & b_i \\
& \forall j \in \{1, \dots, N\} & x_j & \geq & 0 \\
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How to solve this LP?

later...

We can assume that each item has size at least $1/\mathrm{SIZE}(I)$.

Harmonic Grouping

- Sort items according to size (monotonically decreasing).
- Process items in this order; close the current group if size of items in the group is at least 2 (or larger). Then open new group.
- I.e., G_1 is the smallest cardinality set of largest items s.t. total size sums up to at least 2. Similarly, for $G_2, \ldots, G_{r-1}$.
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From the grouping we obtain instance I' as follows:

- Round all items in a group to the size of the largest group member.
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The number of different sizes in I' is at most $\mathrm{SIZE}(I)/2$.

- Each group that survives (recall that G_1 and G_r are deleted) has total size at least 2.
- Hence, the number of surviving groups is at most $\mathrm{SIZE}(I)/2$.
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- Each group that survives (recall that G_1 and G_r are deleted) has total size at least 2.
- Hence, the number of surviving groups is at most $\mathrm{SIZE}(I)/2$.
- All items in a group have the same size in I'.

Lemma 31

The total size of deleted items is at most $\mathcal{O}(\log(\mathrm{SIZE}(I)))$.

- The total size of items in G_1 and G_r is at most 6 as a group has total size at most 3.
- Consider a group G_i that has strictly more items than G_{i-1}.
- It discards $n_i - n_{i-1}$ pieces of total size at most

$$3\frac{n_i - n_{i-1}}{n_i} \le \sum_{j=n_{i-1}+1}^{n_i} \frac{3}{j}$$

since the smallest piece has size at most $3/n_i$.

- Summing over all i that have $n_i > n_{i-1}$ gives a bound of at most

$$\sum_{j=1}^{n_r} \frac{3}{j} \le \mathcal{O}(\log(\mathrm{SIZE}(I)))$$

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Algorithm 1 BinPack

1: **if** $\text{SIZE}(I) < 10$ **then**
2: pack remaining items greedily
3: Apply harmonic grouping to create instance I'; pack discarded items in at most $\mathcal{O}(\log(\text{SIZE}(I)))$ bins.
4: Let x be optimal solution to configuration LP
5: Pack $\lfloor x_j \rfloor$ bins in configuration T_j for all j; call the packed instance I_1.
6: Let I_2 be remaining pieces from I'
7: Pack I_2 via BinPack(I_2)

Analysis

$$\mathrm{OPT}_{\mathrm{LP}}(I_1) + \mathrm{OPT}_{\mathrm{LP}}(I_2) \le \mathrm{OPT}_{\mathrm{LP}}(I') \le \mathrm{OPT}_{\mathrm{LP}}(I)$$

Proof:

- Each piece surviving in I' can be mapped to a piece in I of no lesser size. Hence, $\mathrm{OPT}_{\mathrm{LP}}(I') \le \mathrm{OPT}_{\mathrm{LP}}(I)$
- $\lfloor x_j \rfloor$ is feasible solution for I_1 (even integral).
- $x_j - \lfloor x_j \rfloor$ is feasible solution for I_2.

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Analysis

Each level of the recursion partitions pieces into three types

1. Pieces discarded at this level.
2. Pieces scheduled because they are in I_1.
3. Pieces in I_2 are handed down to the next level.

Pieces of type 2 summed over all recursion levels are packed into at most $\mathrm{OPT}_{\mathrm{LP}}$ many bins.

Pieces of type 1 are packed into at most

$$\mathcal{O}(\log(\mathrm{SIZE}(I))) \cdot L$$

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- The total size of items in I_2 can be at most $\sum_{j=1}^{N} x_j - \lfloor x_j \rfloor$ which is at most the number of non-zero entries in the solution to the configuration LP.

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How to solve the LP?

Let $T_1, \ldots, T_N$ be the sequence of all possible configurations (a configuration T_j has T_{ji} pieces of size s_i).

In total we have b_i pieces of size s_i.

Primal

$$\begin{array}{llll} \min & & \sum_{j=1}^N x_j & \\ \text{s.t.} & \forall i \in \{1 \ldots m\} & \sum_{j=1}^N T_{ji} x_j & \geq b_i \\ & \forall j \in \{1, \ldots, N\} & x_j & \geq 0 \end{array}$$

Dual

$$\begin{array}{llll} \max & & \sum_{i=1}^m y_i b_i & \\ \text{s.t.} & \forall j \in \{1, \ldots, N\} & \sum_{i=1}^m T_{ji} y_i & \leq 1 \\ & \forall i \in \{1, \ldots, m\} & y_i & \geq 0 \end{array}$$

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Separation Oracle

Suppose that I am given variable assignment y for the dual.

How do I find a violated constraint?

I have to find a configuration $T_j = (T_{j1}, \ldots, T_{jm})$ that

- is feasible, i.e.,

$$\sum_{i=1}^{m} T_{ji} \cdot s_i \le 1 \ ,$$

- and has a large profit

$$\sum_{i=1}^{m} T_{ji} y_i > 1$$

But this is the Knapsack problem.

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We have FPTAS for Knapsack. This means if a constraint is violated with $1 + \epsilon' = 1 + \frac{\epsilon}{1-\epsilon}$ we find it, since we can obtain at least $(1 - \epsilon)$ of the optimal profit.

The solution we get is feasible for:

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$$\begin{array}{llrcl} \max & & \sum_{i=1}^{m} y_i b_i & & \\ \text{s.t.} & \forall j \in \{1,\ldots,N\} & \sum_{i=1}^{m} T_{ji} y_i & \le & 1+\epsilon' \\ & \forall i \in \{1,\ldots,m\} & y_i & \ge & 0 \end{array}$$

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If the value of the computed dual solution (which may be infeasible) is z then

$$\mathrm{OPT} \le z \le (1+\epsilon')\mathrm{OPT}$$

How do we get good primal solution (not just the value)?

- The constraints used when computing z certify that the solution is feasible for DUAL'.
- Suppose that we drop all unused constraints in DUAL. We will compute the same solution feasible for DUAL'.
- Let DUAL'' be DUAL without unused constraints.
- The dual to DUAL'' is PRIMAL where we ignore variables for which the corresponding dual constraint has not been used.
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This gives that overall we need at most

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bins.

We can choose $\epsilon' = \frac{1}{\mathrm{OPT}}$ as $\mathrm{OPT} \leq \#\mathrm{items}$ and since we have a fully polynomial time approximation scheme (FPTAS) for knapsack.

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Lemma 32 (Chernoff Bounds)

Let $X_1, \ldots, X_n$ be n independent 0-1 random variables, not necessarily identically distributed. Then for $X = \sum_{i=1}^{n} X_i$ and $\mu = E[X]$, $L \le \mu \le U$, and $\delta > 0$

$$\Pr[X \ge (1+\delta)U] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} ,$$

and

$$\Pr[X \le (1-\delta)L] < \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{L} ,$$

Lemma 33

For $0 \leq \delta \leq 1$ *we have that*

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \leq e^{-U\delta^2/3}$$

and

$$\left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{L} \leq e^{-L\delta^2/2}$$

Proof of Chernoff Bounds

Markovs Inequality:

Let X be random variable taking non-negative values.

Then

$$\Pr[X \geq a] \leq E[X]/a$$

Trivial!

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Hence:

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Hence:

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That's awfully weak :(

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Set $p_i = \Pr[X_i = 1]$. Assume $p_i > 0$ for all i.

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Cool Trick:

$$\Pr[X \geq (1+\delta)U] = \Pr[e^{tX} \geq e^{t(1+\delta)U}]$$

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Set $p_i = \Pr[X_i = 1]$. Assume $p_i > 0$ for all i.

Cool Trick:

$$\Pr[X \geq (1+\delta)U] = \Pr[e^{tX} \geq e^{t(1+\delta)U}]$$

Now, we apply Markov:

$$\Pr[e^{tX} \geq e^{t(1+\delta)U}] \leq \frac{\mathrm{E}[e^{tX}]}{e^{t(1+\delta)U}} .$$

Proof of Chernoff Bounds

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This may be a lot better (!?)

Proof of Chernoff Bounds

$$\mathrm{E}\left[e^{tX}\right]$$

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$$\mathrm{E}\left[e^{tX_i}\right] = (1 - p_i) + p_i e^t = 1 + p_i(e^t - 1) \leq e^{p_i(e^t - 1)}$$

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$$\prod_i \mathrm{E}\left[e^{tX_i}\right] \le \prod_i e^{p_i(e^t-1)} = e^{\sum p_i(e^t-1)}$$

Proof of Chernoff Bounds

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$$\mathrm{E}\left[e^{tX_i}\right] = (1-p_i) + p_i e^t = 1 + p_i(e^t - 1) \le e^{p_i(e^t-1)}$$

$$\prod_i \mathrm{E}\left[e^{tX_i}\right] \le \prod_i e^{p_i(e^t-1)} = e^{\sum p_i(e^t-1)} = e^{(e^t-1)U}$$

Now, we apply Markov:

$$\begin{aligned}\Pr[X \geq (1+\delta)U] &= \Pr[e^{tX} \geq e^{t(1+\delta)U}] \\ &\leq \frac{\mathrm{E}[e^{tX}]}{e^{t(1+\delta)U}}\end{aligned}$$

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We choose $t = \ln(1+\delta)$.

Lemma 34

For $0 \le \delta \le 1$ *we have that*

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \le e^{-U\delta^2/3}$$

and

$$\left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{L} \le e^{-L\delta^2/2}$$

Show:

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \leq e^{-U\delta^2/3}$$

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Take logarithms:

$$U(\delta - (1+\delta)\ln(1+\delta)) \le -U\delta^2/3$$

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True for $\delta = 0$.

Show:

$$\left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{U} \leq e^{-U\delta^2/3}$$

Take logarithms:

$$U(\delta - (1+\delta)\ln(1+\delta)) \leq -U\delta^2/3$$

True for $\delta = 0$. Divide by U and take derivatives:

$$-\ln(1+\delta) \leq -2\delta/3$$

Reason:

As long as derivative of left side is smaller than derivative of right side the inequality holds.

$$f(\delta) := -\ln(1+\delta) + 2\delta/3 \le 0$$

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A convex function ($f''(\delta) \geq 0$) on an interval takes maximum at the boundaries.

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$f(0) = 0$ and $f(1) = -\ln(2) + 2/3 < 0$

For $\delta \geq 1$ we show

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Take logarithms:

$$U(\delta - (1+\delta)\ln(1+\delta)) \leq -U\delta/3$$

True for $\delta = 0$. Divide by U and take derivatives:

$$-\ln(1+\delta) \leq -1/3 \iff \ln(1+\delta) \geq 1/3 \quad \text{(true)}$$

Reason:
As long as derivative of left side is smaller than derivative of right side the inequality holds.

Show:

$$\left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{L} \leq e^{-L\delta^2/2}$$

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True for $\delta = 0$. Divide by L and take derivatives:

$$\ln(1-\delta) \le -\delta$$

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As long as derivative of left side is smaller than derivative of right side the inequality holds.

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This holds for $0 \le \delta < 1$.

Integer Multicommodity Flows

- Given s_i-t_i pairs in a graph.
- Connect each pair by a path such that not too many path use any given edge.

$$
\begin{array}{llrcl}
\min & & & W & \\
\text{s.t.} & \forall i & \sum_{p\in\mathcal{P}_i} x_p & = & 1 \\
 & & \sum_{p:e\in p} x_p & \le & W \\
 & & x_p & \in & \{0,1\}
\end{array}
$$

Integer Multicommodity Flows

Randomized Rounding:
For each i choose one path from the set $\mathcal{P}_i$ at random according to the probability distribution given by the Linear Programming solution.

Theorem 35

If $W^* \geq c \ln n$ for some constant c, then with probability at least $n^{-c/3}$ the total number of paths using any edge is at most $W^* + \sqrt{cW^* \ln n}$.

Theorem 36

With probability at least $n^{-c/3}$ the total number of paths using any edge is at most $W^* + c \ln n$.

Integer Multicommodity Flows

Let X_e^i be a random variable that indicates whether the path for s_i-t_i uses edge e.

Then the number of paths using edge e is $Y_e = \sum_i X_e^i$.

$$E[Y_e] = \sum_i \sum_{p \in \mathcal{P}_i : e \in p} x_p^* = \sum_{p : e \in P} x_p^* \le W^*$$

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Integer Multicommodity Flows

Choose $\delta = \sqrt{(c \ln n)/W^*}$.

Then

$$\Pr[Y_e \geq (1+\delta)W^*] < e^{-W^*\delta^2/3} = \frac{1}{n^{c/3}}$$

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19 MAXSAT

Problem definition:

- n Boolean variables
- m clauses $C_1, \dots, C_m$. For example

$$C_7 = x_3 \vee \bar{x}_5 \vee \bar{x}_9$$

- Non-negative weight w_j for each clause C_j.
- Find an assignment of true/false to the variables sucht that the total weight of clauses that are satisfied is maximum.

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Terminology:

- A variable x_i and its negation $\bar{x}_i$ are called literals.
- Hence, each clause consists of a set of literals (i.e., no duplications: $x_i \vee x_i \vee \bar{x}_j$ is **not** a clause).
- We assume a clause does not contain x_i and $\bar{x}_i$ for any i.
- x_i is called a positive literal while the negation $\bar{x}_i$ is called a negative literal.
- For a given clause C_j the number of its literals is called its length or size and denoted with ℓ_j.
- Clauses of length one are called unit clauses.

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- Clauses of length one are called unit clauses.

MAXSAT: Flipping Coins

Set each x_i independently to true with probability $\frac{1}{2}$ (and, hence, to false with probability $\frac{1}{2}$, as well).

Define random variable X_j with

$$X_j = \begin{cases} 1 & \text{if } C_j \text{ satisfied} \\ 0 & \text{otw.} \end{cases}$$

Then the total weight W of satisfied clauses is given by

$$W = \sum_j w_j X_j$$

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$$E[W]$$

$$E[W] = \sum_j w_j E[X_j]$$

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$$= \sum_j w_j \Pr[C_j \text{ is satisified}]$$

$$
\begin{aligned}
E[W] &= \sum_j w_j E[X_j] \\
&= \sum_j w_j \Pr[C_j \text{ is satisified}] \\
&= \sum_j w_j \Big(1 - \Big(\frac{1}{2}\Big)^{\ell_j}\Big)
\end{aligned}
$$

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&\geq \frac{1}{2} \sum_j w_j \\
&\geq \frac{1}{2} \mathrm{OPT}
\end{aligned}
$$

MAXSAT: LP formulation

- Let for a clause C_j, P_j be the set of positive literals and N_j the set of negative literals.

$$C_j = \bigvee_{j \in P_j} x_i \vee \bigvee_{j \in N_j} \bar{x}_i$$

$$\begin{array}{llrcl} \max & & \sum_j w_j z_j & & \\ \text{s.t.} & \forall j & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) & \geq & z_j \\ & \forall i & y_i & \in & \{0,1\} \\ & \forall j & z_j & \leq & 1 \end{array}$$

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MAXSAT: Randomized Rounding

Set each x_i independently to true with probability y_i (and, hence, to false with probability $(1 - y_i)$).

Lemma 37 (Geometric Mean ≤ Arithmetic Mean)

For any nonnegative $a_1, \dots, a_k$

$$\left(\prod_{i=1}^{k} a_i\right)^{1/k} \le \frac{1}{k}\sum_{i=1}^{k} a_i$$

Definition 38

A function f on an interval I is concave if for any two points s and r from I and any $\lambda \in [0,1]$ we have

$$f(\lambda s + (1-\lambda)r) \geq \lambda f(s) + (1-\lambda)f(r)$$

Lemma 39

Let f be a concave function on the interval $[0,1]$, with $f(0) = a$ and $f(1) = a + b$. Then

$$\begin{aligned} f(\lambda) &= f((1-\lambda)0 + \lambda 1) \\ &\geq (1-\lambda)f(0) + \lambda f(1) \\ &= a + \lambda b \end{aligned}$$

for $\lambda \in [0,1]$.

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$$f(\lambda s + (1-\lambda)r) \geq \lambda f(s) + (1-\lambda)f(r)$$

Lemma 39

Let f be a concave function on the interval $[0,1]$, with $f(0) = a$ and $f(1) = a + b$. Then

$$\begin{aligned} f(\lambda) &= f((1-\lambda)0 + \lambda 1) \\ &\geq (1-\lambda)f(0) + \lambda f(1) \\ &= a + \lambda b \end{aligned}$$

for $\lambda \in [0,1]$.

$\Pr[C_j \text{ not satisfied}]$

$$\Pr[C_j \text{ not satisfied}] = \prod_{i \in P_j} (1 - y_i) \prod_{i \in N_j} y_i$$

$$\Pr[C_j \text{ not satisfied}] = \prod_{i\in P_j}(1-y_i)\prod_{i\in N_j} y_i$$

$$\leq \left[\frac{1}{\ell_j}\left(\sum_{i\in P_j}(1-y_i) + \sum_{i\in N_j} y_i\right)\right]^{\ell_j}$$

$$
\begin{aligned}
\Pr[C_j \text{ not satisfied}] &= \prod_{i\in P_j}(1-y_i)\prod_{i\in N_j} y_i \\
&\le \left[\frac{1}{\ell_j}\left(\sum_{i\in P_j}(1-y_i)+\sum_{i\in N_j} y_i\right)\right]^{\ell_j} \\
&= \left[1-\frac{1}{\ell_j}\left(\sum_{i\in P_j} y_i+\sum_{i\in N_j}(1-y_i)\right)\right]^{\ell_j}
\end{aligned}
$$

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\Pr[C_j \text{ not satisfied}] &= \prod_{i\in P_j}(1-y_i)\prod_{i\in N_j} y_i \\
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&= \left[1-\frac{1}{\ell_j}\left(\sum_{i\in P_j} y_i+\sum_{i\in N_j}(1-y_i)\right)\right]^{\ell_j} \\
&\le \left(1-\frac{z_j}{\ell_j}\right)^{\ell_j} .
\end{aligned}$$

The function $f(z) = 1 - (1 - \frac{z}{\ell})^{\ell}$ is concave. Hence,

$$\Pr[C_j \text{ satisfied}]$$

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$$\geq \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] \cdot z_j \ .$$

The function $f(z) = 1 - (1 - \frac{z}{\ell})^{\ell}$ is concave. Hence,

$$\begin{aligned}\Pr[C_j \text{ satisfied}] &\geq 1 - \left(1 - \frac{z_j}{\ell_j}\right)^{\ell_j} \\ &\geq \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] \cdot z_j \enspace .\end{aligned}$$

$f''(z) = -\frac{\ell-1}{\ell}\left[1 - \frac{z}{\ell}\right]^{\ell-2} \leq 0$ for $z \in [0,1]$. Therefore, f is concave.

$E[W]$

$$E[W] = \sum_j w_j \Pr[C_j \text{ is satisfied}]$$

$$\begin{aligned} E[W] &= \sum_j w_j \Pr[C_j \text{ is satisfied}] \\ &\geq \sum_j w_j z_j \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] \end{aligned}$$

$$
\begin{aligned}
E[W] &= \sum_j w_j \Pr[C_j \text{ is satisfied}] \\
&\geq \sum_j w_j z_j \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] \\
&\geq \left(1 - \frac{1}{e}\right) \mathrm{OPT} \ .
\end{aligned}
$$

MAXSAT: The better of two

Theorem 40
Choosing the better of the two solutions given by randomized rounding and coin flipping yields a $\frac{3}{4}$-approximation.

Let W_1 be the value of randomized rounding and W_2 the value obtained by coin flipping.

$$E[\max\{W_1, W_2\}]$$

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$$
\begin{aligned}
E[\max\{W_1, W_2\}] \\
&\geq E[\tfrac{1}{2}W_1 + \tfrac{1}{2}W_2] \\
&\geq \frac{1}{2}\sum_j w_j z_j\left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] + \frac{1}{2}\sum_j w_j\left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right)
\end{aligned}
$$

Let W_1 be the value of randomized rounding and W_2 the value obtained by coin flipping.

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\begin{aligned}
E[\max\{W_1, W_2\}] & \\
&\geq E[\tfrac{1}{2}W_1 + \tfrac{1}{2}W_2] \\
&\geq \frac{1}{2}\sum_j w_j z_j \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right] + \frac{1}{2}\sum_j w_j \left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right) \\
&\geq \sum_j w_j z_j \Bigg[\underbrace{\frac{1}{2}\left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right) + \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right)}_{\geq \frac{3}{4} \text{ for all integers}}\Bigg]
\end{aligned}
$$

Let W_1 be the value of randomized rounding and W_2 the value obtained by coin flipping.

$$
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&E[\max\{W_1, W_2\}] \\
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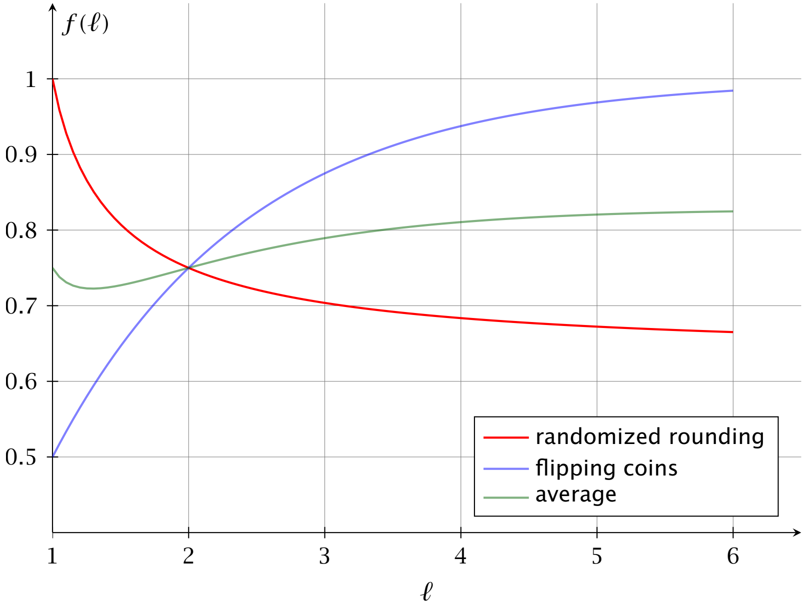
&\qquad \geq \sum_j w_j z_j \Bigg[\underbrace{\frac{1}{2}\left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right) + \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right)}_{\geq \frac{3}{4} \text{ for all integers}}\Bigg] \\
&\qquad \geq \frac{3}{4}\mathrm{OPT}
\end{aligned}
$$


$f(\ell)$
1
0.9
0.8
0.7
0.6
0.5
1
2
3
4
5
6
ℓ
randomized rounding
flipping coins
average

MAXSAT: Nonlinear Randomized Rounding

So far we used linear randomized rounding, i.e., the probability that a variable is set to 1/true was exactly the value of the corresponding variable in the linear program.

We could define a function $f : [0,1] \rightarrow [0,1]$ and set x_i to true with probability $f(y_i)$.

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MAXSAT: Nonlinear Randomized Rounding

Let $f : [0,1] \to [0,1]$ be a function with

$$1 - 4^{-x} \le f(x) \le 4^{x-1}$$

Theorem 41

Rounding the LP-solution with a function f of the above form gives a $\frac{3}{4}$-approximation.

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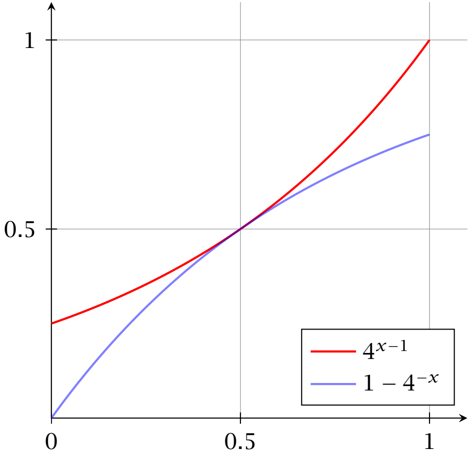
1
0.5
0
0.5
1
4^{x-1}
$1-4^{-x}$

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&\le 4^{-z_j}
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Can we do better?

Not if we compare ourselves to the value of an optimum LP-solution.

Definition 42 (Integrality Gap)

The integrality gap for an ILP is the worst-case ratio over all instances of the problem of the value of an optimal IP-solution to the value of an optimal solution to its linear programming relaxation.

Note that the integrality is less than one for maximization problems and larger than one for minimization problems (of course, equality is possible).

Note that an integrality gap only holds for one specific ILP formulation.

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Lemma 43

Our ILP-formulation for the MAXSAT problem has integrality gap at most $\frac{3}{4}$.

$$
\begin{array}{lllcl}
\max & & \sum_j w_j z_j & & \\
\text{s.t.} & \forall j & \sum_{i\in P_j} y_i + \sum_{i\in N_j}(1-y_i) & \geq & z_j \\
 & \forall i & y_i & \in & \{0,1\} \\
 & \forall j & z_j & \leq & 1
\end{array}
$$

Consider: $(x_1 \vee x_2) \wedge (\bar{x}_1 \vee x_2) \wedge (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_2)$

- any solution can satisfy at most 3 clauses
- we can set $y_1 = y_2 = 1/2$ in the LP; this allows to set $z_1 = z_2 = z_3 = z_4 = 1$
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Repetition: Primal Dual for Set Cover

Primal Relaxation:

$$\begin{array}{llrcl} \min & & \sum_{i=1}^{k} w_i x_i & & \\ \text{s.t.} & \forall u \in U & \sum_{i:u \in S_i} x_i & \geq & 1 \\ & \forall i \in \{1,\ldots,k\} & x_i & \geq & 0 \end{array}$$

Dual Formulation:

$$\begin{array}{llrcl} \max & & \sum_{u \in U} y_u & & \\ \text{s.t.} & \forall i \in \{1,\ldots,k\} & \sum_{u:u \in S_i} y_u & \leq & w_i \\ & & y_u & \geq & 0 \end{array}$$

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Algorithm:

- Start with $y = 0$ (feasible dual solution).
  Start with $x = 0$ (integral primal solution that may be infeasible).
- While x not feasible
  - Identify an element e that is not covered in current primal integral solution.
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- Hence our cost is

$$\sum_j w_j = \sum_j \sum_{e \in S_j} y_e = \sum_e |\{j : e \in S_j\}| \cdot y_e \le f \cdot \sum_e y_e \le f \cdot \mathrm{OPT}$$

Note that the constructed pair of primal and dual solution fulfills primal slackness conditions.

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This means

$$x_j > 0 \Rightarrow \sum_{e \in S_j} y_e = w_j$$

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If we would also fulfill dual slackness conditions

$$y_e > 0 \Rightarrow \sum_{j : e \in S_j} x_j = 1$$

then the solution would be optimal!!!

We don’t fulfill these constraint but we fulfill an approximate version:

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This is sufficient to show that the solution is an f-approximation.

Suppose we have a primal/dual pair

$$\begin{array}{llrcl} \min & & \sum_j c_j x_j & & \\ \text{s.t.} & \forall i & \sum_{j:} a_{ij} x_j & \geq & b_i \\ & \forall j & x_j & \geq & 0 \end{array}$$

$$\begin{array}{llrcl} \max & & \sum_i b_i y_i & & \\ \text{s.t.} & \forall j & \sum_i a_{ij} y_i & \leq & c_j \\ & \forall i & y_i & \geq & 0 \end{array}$$

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min		$\sum_j c_j x_j$		
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max		$\sum_i b_i y_i$		
s.t.	$\forall j$	$\sum_i a_{ij} y_i$	$\leq$	c_j
	$\forall i$	y_i	$\geq$	0

and solutions that fulfill approximate slackness conditions:

$$x_j > 0 \Rightarrow \sum_i a_{ij} y_i \geq \frac{1}{\alpha} c_j$$

$$y_i > 0 \Rightarrow \sum_j a_{ij} x_j \leq \beta b_i$$

Then

$$\sum_j c_j x_j$$

Then

$$\sum_j c_j x_j$$

primal cost

Then

right hand side of j-th dual constraint

$$\sum_j c_j x_j$$

primal cost

Then

$$\boxed{\sum_j c_j x_j} \leq \alpha \sum_j \left(\sum_i a_{ij} y_i \right) x_j$$

primal cost

Then

$$\sum_j c_j x_j \le \alpha \sum_j \left(\sum_i a_{ij} y_i\right) x_j$$

$$= \alpha \sum_i \left(\sum_j a_{ij} x_j\right) y_i$$

primal cost

Then

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primal cost

$$= \alpha \sum_i \left(\sum_j a_{ij} x_j \right) y_i$$

$$\leq \alpha\beta \cdot \sum_i b_i y_i$$

Then

$$\sum_j c_j x_j \le \alpha \sum_j \left(\sum_i a_{ij} y_i \right) x_j$$

primal cost

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$$\le \alpha\beta \cdot \sum_i b_i y_i$$

dual objective

Feedback Vertex Set for Undirected Graphs

- Given a graph $G = (V, E)$ and non-negative weights $w_v \geq 0$ for vertex $v \in V$.

Feedback Vertex Set for Undirected Graphs

- Given a graph $G = (V, E)$ and non-negative weights $w_v \geq 0$ for vertex $v \in V$.
- Choose a minimum cost subset of vertices s.t. every cycle contains at least one vertex.

We can encode this as an instance of Set Cover

- Each vertex can be viewed as a set that contains some cycles.

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- However, this encoding gives a Set Cover instance of non-polynomial size.

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- Each vertex can be viewed as a set that contains some cycles.
- However, this encoding gives a Set Cover instance of non-polynomial size.
- The $O(\log n)$-approximation for Set Cover does not help us to get a good solution.

Let C denote the set of all cycles (where a cycle is identified by its set of vertices)

Let $\mathcal{C}$ denote the set of all cycles (where a cycle is identified by its set of vertices)

Primal Relaxation:

$$\begin{array}{llrcl}\min & & \sum_v w_v x_v & & \\ \text{s.t.} & \forall C \in \mathcal{C} & \sum_{v \in C} x_v & \geq & 1 \\ & \forall v & x_v & \geq & 0\end{array}$$

Dual Formulation:

$$\begin{array}{llrcl}\max & & \sum_{C \in \mathcal{C}} y_C & & \\ \text{s.t.} & \forall v \in V & \sum_{C: v \in C} y_C & \leq & w_v \\ & \forall C & y_C & \geq & 0\end{array}$$

If we perform the previous dual technique for Set Cover we get the following:

- Start with $x = 0$ and $y = 0$

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$$\sum_v w_v x_v$$

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where S is the set of vertices we choose.

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$$\begin{aligned}\sum_v w_v x_v &= \sum_v \sum_{C:v\in C} y_C x_v \\ &= \sum_{v\in S} \sum_{C:v\in C} y_C \\ &= \sum_C |S\cap C|\cdot y_C\end{aligned}$$

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where S is the set of vertices we choose.

If every cycle is short we get a good approximation ratio, but this is unrealistic.

Algorithm 1 FeedbackVertexSet

1: $y \leftarrow 0$
2: $x \leftarrow 0$
3: **while** exists cycle C in G **do**
4: increase y_C until there is $v \in C$ s.t. $\sum_{C:v\in C} y_C = w_v$
5: $x_v = 1$
6: remove v from G
7: repeatedly remove vertices of degree 1 from G

Idea:

Always choose a short cycle that is not covered. If we always find a cycle of length at most α we get an α-approximation.

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Observation:

For any path P of vertices of degree 2 in G the algorithm chooses at most one vertex from P.

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If we always choose a cycle for which the number of vertices of degree at least 3 is at most α we get a 2α-approximation.

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Theorem 44

In any graph with no vertices of degree 1, there always exists a cycle that has at most $\mathcal{O}(\log n)$ vertices of degree 3 or more. We can find such a cycle in linear time.

This means we have

$$y_C > 0 \Rightarrow |S \cap C| \leq \mathcal{O}(\log n) \ .$$

Primal Dual for Shortest Path

Given a graph $G = (V, E)$ with two nodes $s, t \in V$ and edge-weights $c : E \to \mathbb{R}^+$ find a shortest path between s and t w.r.t. edge-weights c.

$$
\begin{array}{llrcl}
\min & & \sum_e c(e) x_e & & \\
\text{s.t.} & \forall S \in \mathcal{S} & \sum_{e:\delta(S)} x_e & \geq & 1 \\
 & \forall e \in E & x_e & \in & \{0,1\}
\end{array}
$$

Here $\delta(S)$ denotes the set of edges with exactly one end-point in S, and $\mathcal{S} = \{S \subseteq V : s \in S, t \notin S\}$.

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Primal Dual for Shortest Path

The Dual:

$$\begin{array}{llrcl} \max & & \sum_S y_S & & \\ \text{s.t.} & \forall e \in E & \sum_{S:e\in\delta(S)} y_S & \le & c(e) \\ & \forall S \in \mathcal{S} & y_S & \ge & 0 \end{array}$$

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Primal Dual for Shortest Path

We can interpret the value y_S as the width of a moat surounding the set S.

Each set can have its own moat but all moats must be disjoint.

An edge cannot be shorter than all the moats that it has to cross.

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An edge cannot be shorter than all the moats that it has to cross.

Algorithm 1 PrimalDualShortestPath

$y \leftarrow 0$
$F \leftarrow \emptyset$
while there is no s-t path in (V, F) **do**
    Let C be the connected component of (V, F) containing s
    Increase y_C until there is an edge $e' \in \delta(C)$ such that $\sum_{S: e' \in \delta(S)} y_S = c(e')$.
    $F \leftarrow F \cup \{e'\}$
Let P be an s-t path in (V, F)
return P

Lemma 45

At each point in time the set F forms a tree.

Proof:

- In each iteration we take the current connected component from (V, F) that contains s (call this component C) and add some edge from $\delta(C)$ to F.
- Since, at most one end-point of the new edge is in C the edge cannot close a cycle.

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$$\sum_{e \in P} c(e)$$

$$\sum_{e\in P} c(e) = \sum_{e\in P} \sum_{S:e\in\delta(S)} y_S$$

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If we can show that $y_S > 0$ implies $|P \cap \delta(S)| = 1$ gives

$$\sum_{e\in P} c(e) = \sum_{S} y_S \le \mathrm{OPT}$$

by weak duality.

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Hence, we find a shortest path.

If S contains two edges from P then there must exist a subpath P' of P that starts and ends with a vertex from S (and all interior vertices are not in S).

When we increased y_S, S was a connected component of the set of edges F' that we had chosen till this point.

$F' \cup P'$ contains a cycle. Hence, also the final set of edges contains a cycle.

This is a contradiction.

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Steiner Forest Problem:

Given a graph $G = (V, E)$, together with source-target pairs $s_i, t_i, i = 1, \dots, k$, and a cost function $c : E \to \mathbb{R}^+$ on the edges. Find a subset $F \subseteq E$ of the edges such that for every $i \in \{1, \dots, k\}$ there is a path between s_i and t_i only using edges in F.

$$\begin{array}{llrcl} \min & & \sum_e c(e) x_e & & \\ \text{s.t.} & \forall S \subseteq V : S \in S_i \text{ for some } i & \sum_{e \in \delta(S)} x_e & \geq & 1 \\ & \forall e \in E & x_e & \in & \{0, 1\} \end{array}$$

Here S_i contains all sets S such that $s_i \in S$ and $t_i \notin S$.

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$$
\begin{array}{llrcl}
\max & & \sum_{S\,:\,\exists i \text{ s.t. } S \in \mathcal{S}_i} y_S & & \\
\text{s.t.} & \forall e \in E & \sum_{S: e \in \delta(S)} y_S & \leq & c(e) \\
& & y_S & \geq & 0
\end{array}
$$

The difference to the dual of the shortest path problem is that we have many more variables (sets for which we can generate a moat of non-zero width).

Algorithm 1 FirstTry

```
y ← 0
F ← ∅
while not all s_i-t_i pairs connected in F do
    Let C be some connected component of (V, F) such that |C ∩ {s_i, t_i}| = 1 for some i.
    Increase y_C until there is an edge e' ∈ δ(C) s.t. ∑_{S∈S_i : e'∈δ(S)} y_S = c_{e'}
    F ← F ∪ {e'}
return ⋃_i P_i
```

$$\sum_{e \in F} c(e)$$

$$\sum_{e\in F} c(e) = \sum_{e\in F}\sum_{S:e\in\delta(S)} y_S$$

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$$\sum_{e\in F} c(e) = \sum_{e\in F} \sum_{S:e\in\delta(S)} y_S = \sum_S |\delta(S)\cap F| \cdot y_S \enspace .$$

If we show that $y_S > 0$ implies that $|\delta(S)\cap F| \le \alpha$ we are in good shape.

However, this is not true:

- Take a complete graph on $k+1$ vertices $v_0, v_1, \dots, v_k$.

$$\sum_{e\in F} c(e) = \sum_{e\in F}\sum_{S:e\in\delta(S)} y_S = \sum_S |\delta(S)\cap F|\cdot y_S \ .$$

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- The first component C could be $\{v_0\}$.

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- The i-th pair is v_0-v_i.
- The first component C could be $\{v_0\}$.
- We only set $y_{\{v_0\}} = 1$. All other dual variables stay 0.
- The final set F contains all edges $\{v_0, v_i\}$, $i = 1, \dots, k$.
- $y_{\{v_0\}} > 0$ but $|\delta(\{v_0\})\cap F| = k$.

Algorithm 1 SecondTry

$y \leftarrow 0$; $F \leftarrow \emptyset$; $\ell \leftarrow 0$

while not all s_i-t_i pairs connected in F **do**

    $\ell \leftarrow \ell + 1$

    Let $\mathcal{C}$ be set of all connected components C of (V, F) such that $|C \cap \{s_i, t_i\}| = 1$ for some i.

    Increase y_C for all $C \in \mathcal{C}$ uniformly until for some edge $e_\ell \in \delta(C')$, $C' \in \mathcal{C}$ s.t. $\sum_{S: e_\ell \in \delta(S)} y_S = c_{e_\ell}$

    $F \leftarrow F \cup \{e_\ell\}$

$F' \leftarrow F$

for $k \leftarrow \ell$ downto 1 **do** // reverse deletion

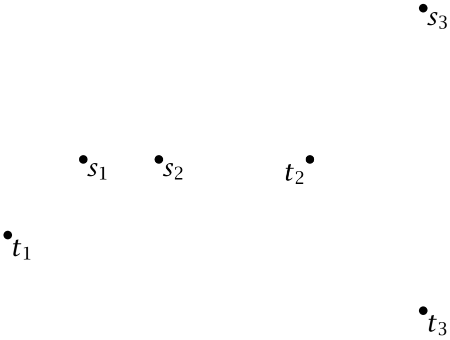
    if $F' - e_k$ is feasible solution **then**

        remove e_k from F'

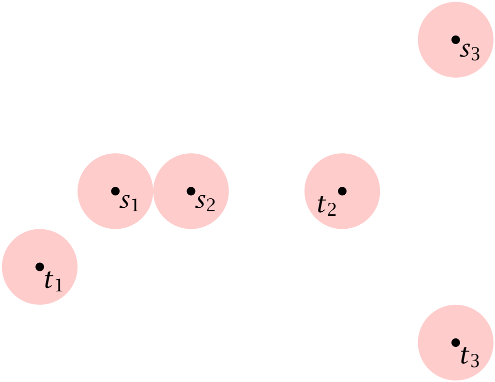
return F'

The reverse deletion step is not strictly necessary this way. It would also be sufficient to simply delete all unnecessary edges in any order.

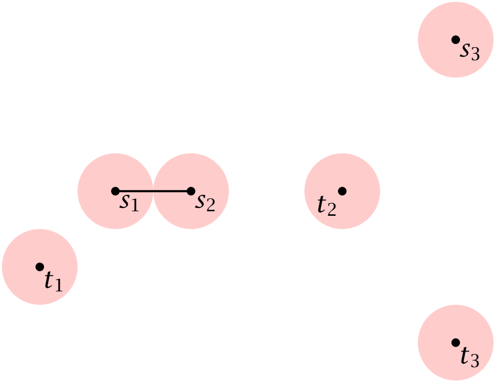
Example



Example



Example



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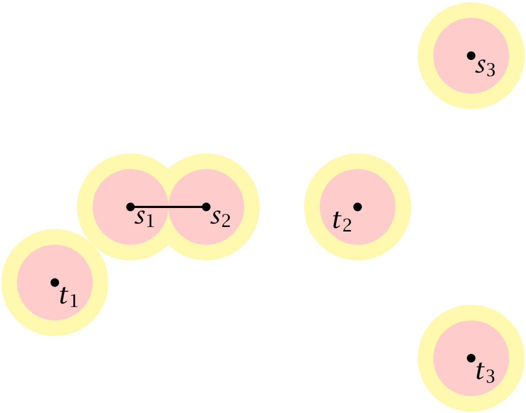

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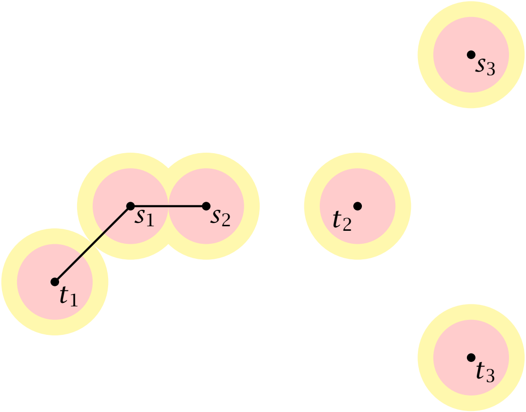

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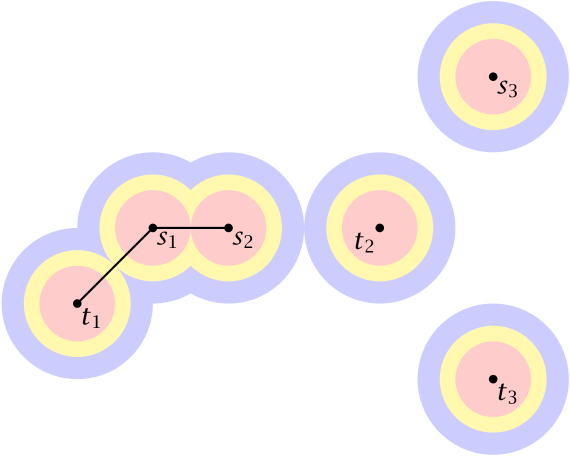

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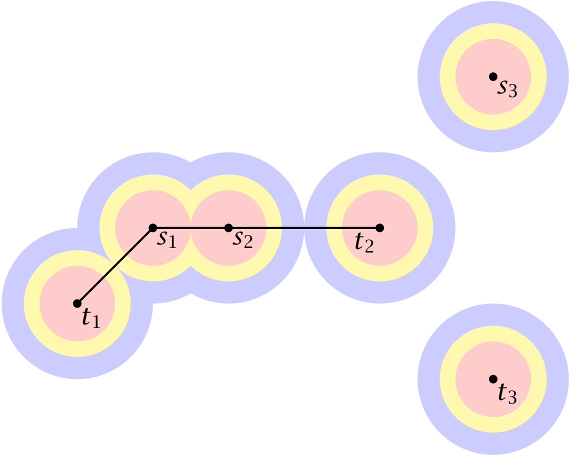

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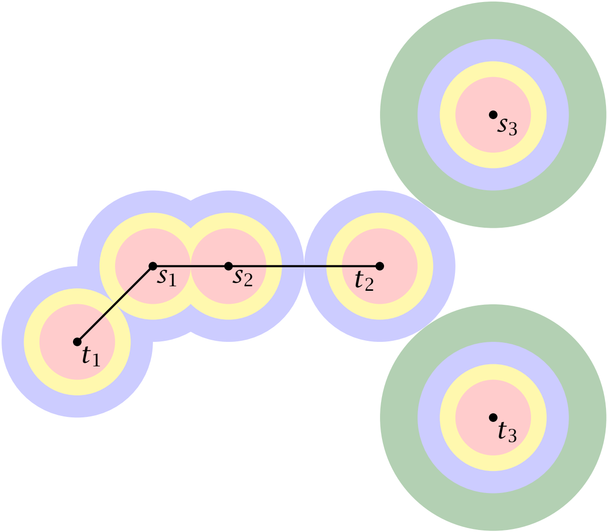

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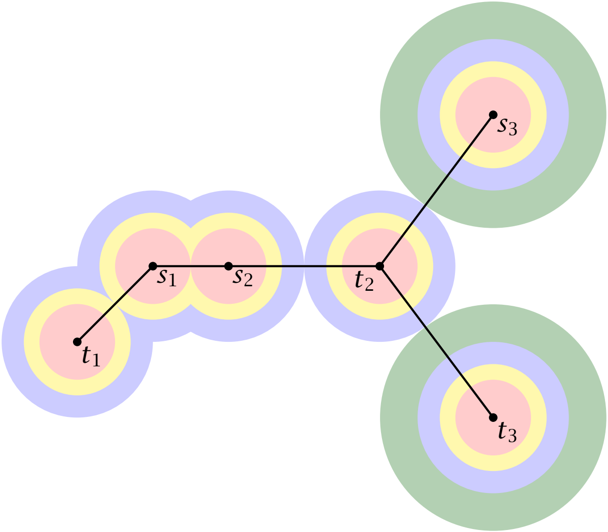

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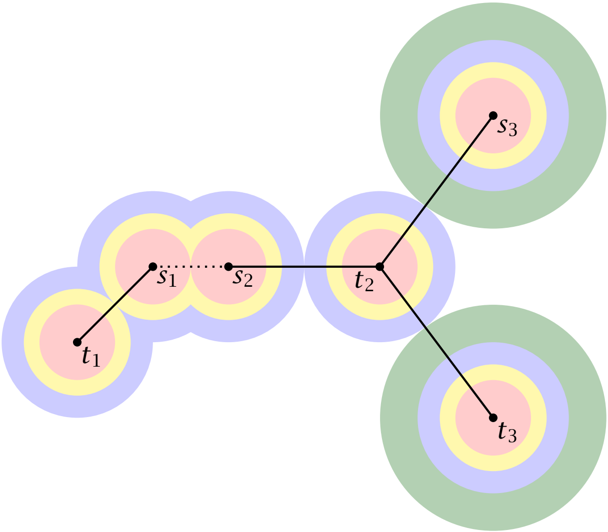

Lemma 46

For any $\mathcal{C}$ in any iteration of the algorithm

$$\sum_{C \in \mathcal{C}} |\delta(C) \cap F'| \leq 2|\mathcal{C}|$$

This means that the number of times a moat from $\mathcal{C}$ is crossed in the final solution is at most twice the number of moats.

Proof: later...

$$\sum_{e\in F'} c_e = \sum_{e\in F'} \sum_{S:e\in\delta(S)} y_S = \sum_S |F'\cap\delta(S)|\cdot y_S \enspace .$$

We want to show that

$$\sum_S |F'\cap\delta(S)|\cdot y_S \le 2\sum_S y_S$$

- In the i-th iteration the increase of the left hand side is

$$\epsilon \sum_{C\in\mathcal{C}} |F'\cap\delta(C)|$$

and the increase of the right hand side is $2\epsilon|\mathcal{C}|$.

- Hence, by the previous lemma the inequality holds after the iteration if it holds in the beginning of the iteration.

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$$\sum_S |F' \cap \delta(S)| \cdot y_S \le 2 \sum_S y_S$$

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$$\epsilon \sum_{S\in A} |F' \cap \delta(S)|$$

and the increase of the right hand side is $2\epsilon|A|$.

Hence, by the previous lemma the inequality holds after the iteration if it holds in the beginning of the iteration.

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We want to show that

$$\sum_S |F' \cap \delta(S)| \cdot y_S \le 2 \sum_S y_S$$

- In the i-th iteration the increase of the left-hand side is

$$\epsilon \sum_{C\in C} |F' \cap \delta(C)|$$

  and the increase of the right hand side is $2\epsilon|C|$.
- Hence, by the previous lemma the inequality holds after the iteration if it holds in the beginning of the iteration.

Lemma 47

For any set of connected components $\mathcal{C}$ in any iteration of the algorithm

$$\sum_{C\in\mathcal{C}} |\delta(C)\cap F'| \le 2|\mathcal{C}|$$

Proof:

- At any point during the algorithm the set of edges forms a forest (why?).
- Fix iteration i. e_i is the set we add to F. Let F_i be the set of edges in F at the beginning of the iteration.
- Let $H = F' - F_i$.
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- Contract all edges in F_i into single vertices V'.
- We can consider the forest H on the set of vertices V'.
- Let $\deg(v)$ be the degree of a vertex $v \in V'$ within this forest.
- Color a vertex $v \in V'$ red if it corresponds to a component from C (an active component). Otw. color it blue. (Let B the set of blue vertices (with non-zero degree) and R the set of red vertices)
- We have

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  - But then it must be a red node.

21 Cuts & Metrics

Shortest Path

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\begin{array}{llrcl}
\min & & \sum_e c(e) x_e & & \\
\text{s.t.} & \forall S \in \mathcal{S} & \sum_{e \in \delta(S)} x_e & \geq & 1 \\
& \forall e \in E & x_e & \in & \{0,1\}
\end{array}
$$

$\mathcal{S}$ is the set of subsets that separate s from t.

The Dual:

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The Separation Problem for the Shortest Path LP is the Minimum Cut Problem.

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Observations:

Suppose that ℓ_e-values are solution to Minimum Cut LP.

- We can view ℓ_e as defining the length of an edge.
- Define $d(u,v) = \min_{\text{path } P \text{ btw. } u \text{ and } v} \sum_{e\in P} \ell_e$ as the Shortest Path Metric induced by ℓ_e.
- We have $d(u,v) = \ell_e$ for every edge $e = (u,v)$, as otw. we could reduce ℓ_e without affecting the distance between s and t.

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d is not a metric on V but a semimetric as two nodes u and v could have distance zero.

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How do we round the LP?

- Let $B(s,r)$ be the ball of radius r around s (w.r.t. metric d). Formally:

$$B = \{v \in V \mid d(s,v) \le r\}$$

- For $0 \le r < 1$, $B(s,r)$ is an s-t-cut.

Which value of r should we choose? choose randomly!!!

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choose r u.a.r. (uniformly at random) from interval $[0,1)$

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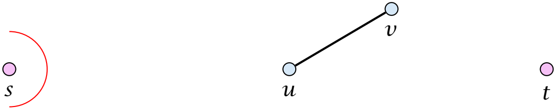
What is the probability that an edge (u, v) is in the cut?



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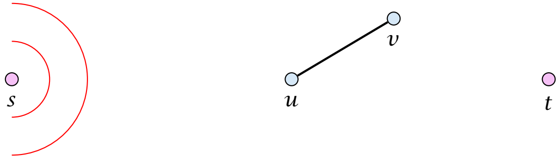

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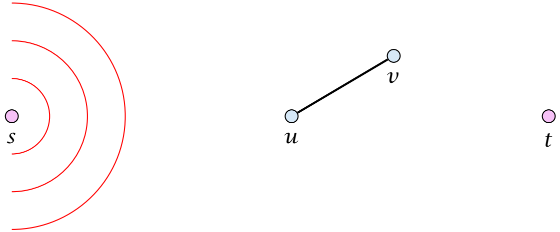

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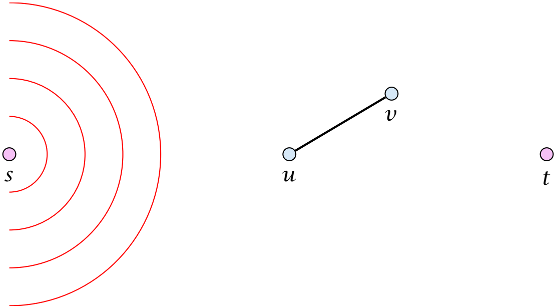

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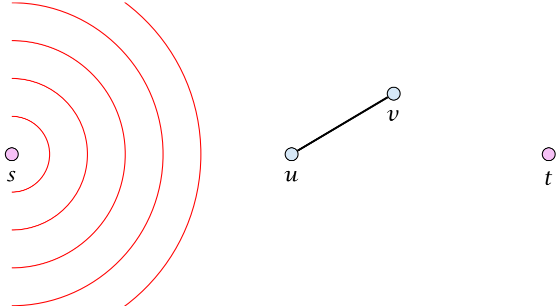

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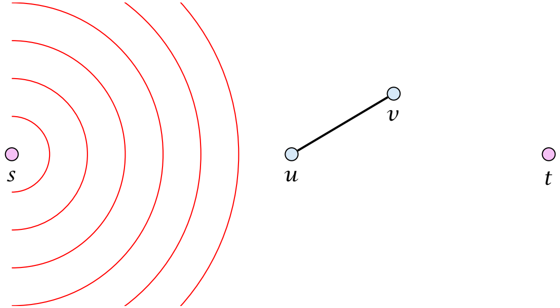

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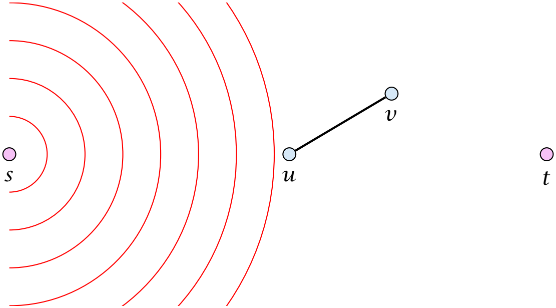

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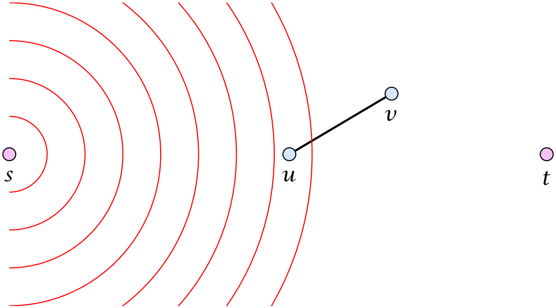

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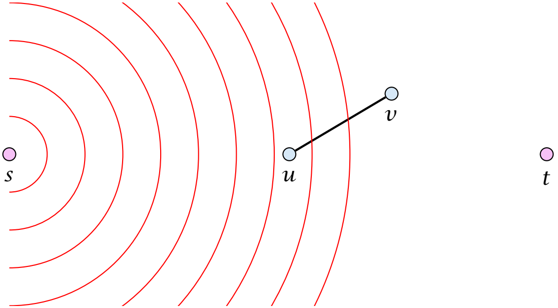

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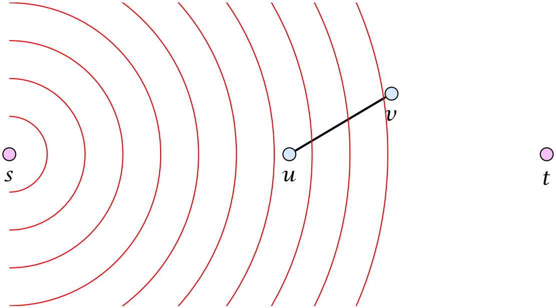

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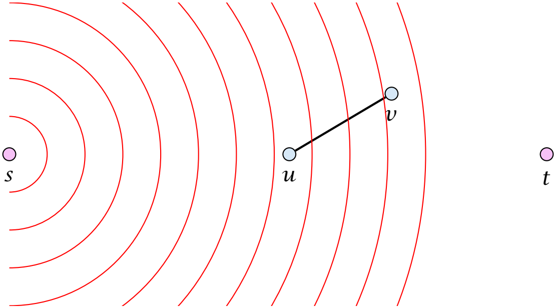

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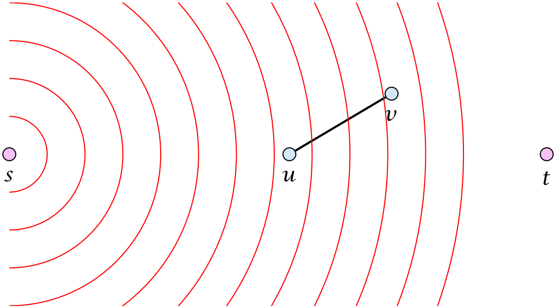

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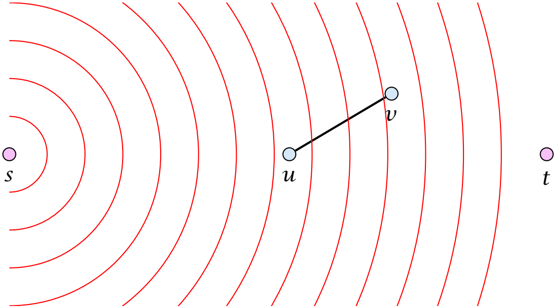

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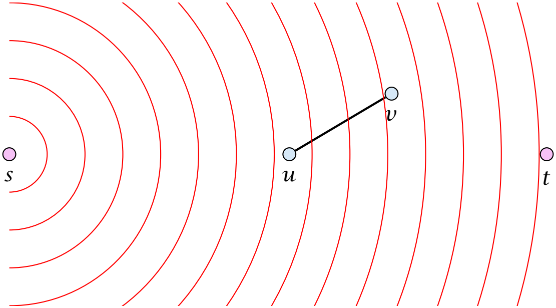

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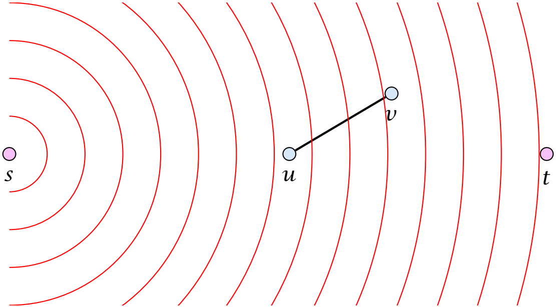

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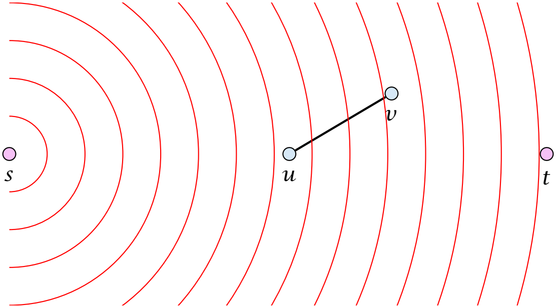

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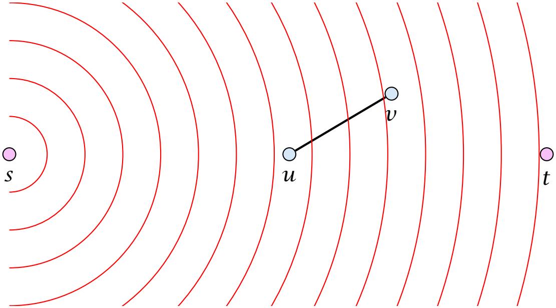

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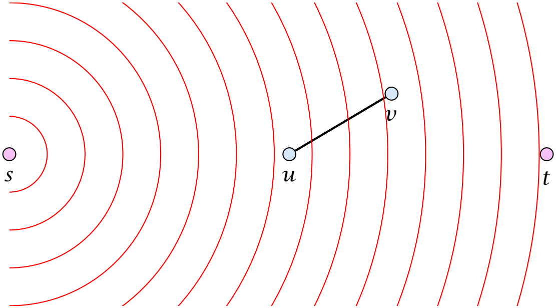

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$$\le \ell_e$$

What is the expected size of a cut?

$$\mathrm{E}[\text{size of cut}] = \mathrm{E}[\textstyle\sum_e c(e)\Pr[e \text{ is cut}]]$$
$$\le \textstyle\sum_e c(e)\ell_e$$

On the other hand:

$$\textstyle\sum_e c(e)\ell_e \le \text{size of mincut}$$

as the ℓ_e are the solution to the Mincut LP *relaxation*.

Hence, our rounding gives an optimal solution.

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Minimum Multicut:
Given a graph $G = (V, E)$, together with source-target pairs s_i, t_i, $i = 1, \dots, k$, and a capacity function $c : E \to \mathbb{R}^+$ on the edges. Find a subset $F \subseteq E$ of the edges such that all s_i-t_i pairs lie in different components in $G = (V, E \setminus F)$.

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Re-using the analysis for the single-commodity case is difficult.

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- Assume for simplicity that all edge-length ℓ_e are multiples of $\delta \ll 1$.
- Replace the graph G by a graph G', where an edge of length ℓ_e is replaced by ℓ_e/δ edges of length δ.
- Let $B(s_i, z)$ be the ball in G' that contains nodes v with distance $d(s_i, v) \le z\delta$.

Algorithm 1 RegionGrowing(s_i, p)

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z ← 0
repeat
    flip a coin (Pr[heads] = p)
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Algorithm 1 Multicut(G')

while $\exists s_i$-t_i pair in G' **do**
  $C \leftarrow$ RegionGrowing(s_i, p)
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- probability of cutting an edge is only p
- a source either does not reach an edge during Region Growing; then it is not cut
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Note: success means all source-target pairs separated

We assume $k \geq 2$.

If we are not successful we simply perform a trivial k-approximation.

This only increases the expected cost by at most $\frac{1}{k^2} \cdot k\mathrm{OPT} \le \mathrm{OPT}/k$.

Hence, our final cost is $\mathcal{O}(\ln k) \cdot \mathrm{OPT}$ in expectation.

Facility Location

Given a set L of (possible) locations for placing facilities and a set D of customers together with cost functions $s : D \times L \to \mathbb{R}^+$ and $o : L \to \mathbb{R}^+$ find a set of facility locations F together with an assignment $\phi : D \to F$ of customers to open facilities such that

$$\sum_{f \in F} o(f) + \sum_{c} s(c, \phi(c))$$

is minimized.

In the metric facility location problem we have

$$s(c,f) \le s(c,f') + s(c',f) + s(c',f') \enspace .$$

Facility Location

Integer Program

$$
\begin{array}{llrcl}
\min & & \sum_{i\in F} f_i y_i + \sum_{i\in F}\sum_{j\in D} c_{ij}x_{ij} & & \\
\text{s.t.} & \forall j \in D & \sum_{i\in F} x_{ij} & = & 1 \\
 & \forall i \in F, j \in D & x_{ij} & \le & y_i \\
 & \forall i \in F, j \in D & x_{ij} & \in & \{0,1\} \\
 & \forall i \in F & y_i & \in & \{0,1\}
\end{array}
$$

As usual we get an LP by relaxing the integrality constraints.

Facility Location

Dual Linear Program

$$
\begin{array}{llrcl}
\max & & \sum_{j\in D} v_j & & \\
\textbf{s.t.} & \forall i \in F & \sum_{j\in D} w_{ij} & \leq & f_i \\
& \forall i \in F, j \in D & v_j - w_{ij} & \leq & c_{ij} \\
& \forall i \in F, j \in D & w_{ij} & \geq & 0
\end{array}
$$

Definition 48

Given an LP solution (x^*, y^*) we say that facility i neighbours client j if $x_{ij} > 0$. Let $N(j) = \{i \in F : x^*_{ij} > 0\}$.

Lemma 49
If (x^*, y^*) is an optimal solution to the facility location LP and (v^*, w^*) is an optimal dual solution, then $x^*_{ij} > 0$ implies $c_{ij} \le v^*_j$.

Follows from slackness conditions.

Suppose we open set $S \subseteq F$ of facilities s.t. for all clients we have $S \cap N(j) \neq \emptyset$.

Then every client j has a facility i s.t. assignment cost for this client is at most $c_{ij} \leq v_j^*$.

Hence, the total assignment cost is

$$\sum_j c_{i_j j} \leq \sum_j v_j^* \leq \mathrm{OPT} \enspace ,$$

where i_j is the facility that client j is assigned to.

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Problem: Facility cost may be huge!

Suppose we can partition a subset $F' \subseteq F$ of facilities into neighbour sets of some clients. I.e.

$$F' = \biguplus_k N(j_k)$$

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Now in each set $N(j_k)$ we open the cheapest facility. Call it f_{i_k}.

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Facility cost is at most the facility cost in an optimum solution.

Problem: so far clients $j_1, j_2, \ldots$ have a neighboring facility. What about the others?

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Let $N^2(j)$ denote all neighboring clients of the neighboring facilities of client j.

Note that $N(j)$ is a set of facilities while $N^2(j)$ is a set of clients.

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Algorithm 1 FacilityLocation

```
C ← D// unassigned clients
k ← 0
while C ≠ 0 do
    k ← k + 1
    choose j_k ∈ C that minimizes v_j^*
    choose i_k ∈ N(j_k) as cheapest facility
    assign j_k and all unassigned clients in N^2(j_k) to i_k
    C ← C − {j_k} − N^2(j_k)
```

Facility cost of this algorithm is at most OPT because the sets $N(j_k)$ are disjoint.

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Total assignment cost:

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$$c_{i\ell}$$

Facility cost of this algorithm is at most OPT because the sets $N(j_k)$ are disjoint.

Total assignment cost:

- Fix k; set $j = j_k$ and $i = i_k$. We know that $c_{ij} \le v_j^*$.
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Summing this over all facilities gives that the total assignment cost is at most $3 \cdot \mathrm{OPT}$. Hence, we get a 4-approximation.

In the above analysis we use the inequality

$$\sum_{i \in F} f_i y_i^* \leq \text{OPT} \ .$$

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We know something stronger namely

$$\sum_{i \in F} f_i y_i^* + \sum_{i \in F} \sum_{j \in D} c_{ij} x_{ij}^* \leq \mathrm{OPT} \ .$$

Observation:

- Suppose when choosing a client j_k, instead of opening the cheapest facility in its neighborhood we choose a random facility according to $x^*_{ij_k}$.
- Then we incur connection cost

$$\sum_i c_{ij_k} x^*_{ij_k}$$

for client j_k. (In the previous algorithm we estimated this by $v^*_{j_k}$).

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What will our facility cost be?

We only try to open a facility once (when it is in neighborhood of some j_k). (recall that neighborhoods of different j_k's are disjoint).

We open facility i with probability $x_{ij_k} \leq y_i$ (in case it is in some neighborhood; otw. we open it with probability zero).

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Algorithm 1 FacilityLocation

$C \leftarrow D$// unassigned clients
$k \leftarrow 0$
while $C \neq 0$ **do**
  $k \leftarrow k + 1$
  choose $j_k \in C$ that minimizes $v_j^* + C_j^*$
  choose $i_k \in N(j_k)$ according to probability x_{ij_k}.
  assign j_k and all unassigned clients in $N^2(j_k)$ to i_k
  $C \leftarrow C - \{j_k\} - N^2(j_k)$

Total assignment cost:

- Fix k; set $j = j_k$.
- Let $\ell \in N^2(j)$ and h (one of) its neighbour(s) in $N(j)$.
- If we assign a client ℓ to the same facility as i we pay at most

$$\sum_i c_{ij} x^*_{ij_k} + c_{hj} + c_{h\ell} \le C^*_j + v^*_j + v^*_\ell \le C^*_\ell + 2v^*_\ell$$

Summing this over all clients gives that the total assignment cost is at most

$$\sum_j C^*_j + \sum_j 2v^*_j \le \sum_j C^*_j + 2\mathrm{OPT}$$

Hence, it is at most $2\mathrm{OPT}$ plus the total assignment cost in an optimum solution.

Adding the facility cost gives a 3-approximation.

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