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Randomized Algorithms

Exercise Sheet 12

Due: January 26, 2015

Exercise 1: (10 points)
We are given the following algorithm for constructing a random boolean CNF formula f with n boolean variables and m clauses. In a formula generated by this algorithm, the same clause might appear more than once and each clause contains exactly 3 literals which correspond to 3 different variables.

- for $j = 1, 2, \ldots, m$
  pick independently and uniformly at random one among all possible $8\binom{n}{3}$ clauses

Initially, compute the expected number of satisfying truth assignments as a function of n and m. Next, show that the probability that f is satisfiable (i.e. there exists at least one truth assignment) is $O(0.9^n)$ when $m = 6n$.

Exercise 2: (10 points)
Consider a boolean CNF formula f consisting of n variables and m clauses. Let k_i be the number of literals of the i-th clause, for $i = 1, 2, \ldots, m$. Recall the randomized algorithm presented in class which assigns a value true or false to each variable independently and uniformly at random. The expected number of satisfied clauses by a truth assignment obtained with this algorithm is at least $\sum_{i=1}^{m}(1 - \frac{1}{2^{k_i}})$. Derandomize the algorithm by using the method of conditional expectations.

Exercise 3: (10 points)
Let $G = (V, E)$ be an undirected graph with n vertices and let d_i be the degree of vertex $i \in V$. Given a permutation σ of the vertices in V, we construct a subset $S(\sigma) \subseteq V$ of the vertices as follows.

- for each vertex $i \in V$
  $i \in S(\sigma)$ if and only if no neighbor j of i precedes i in the permutation σ

Initially, show that $S(\sigma)$ is an independent set of G. Next, propose a randomized algorithm for producing σ so that the expected size of the independent set $S(\sigma)$ is $\sum_{i=1}^{n} \frac{1}{d_i+1}$. Finally, explain why G has always an independent set of size at least $\sum_{i=1}^{n} \frac{1}{d_i+1}$.

Exercise 4: (10 points)
Let T be a binary search tree in which all the keys are distinct. Consider a leaf x of T and let y be its parent. Show that $key(y)$ is either the smallest key in T larger than $key(x)$ or the largest key in T smaller than $key(x)$.