7.5 (a, b)-trees

Definition 1

For $b \geq 2a - 1$ an (a, b)-tree is a search tree with the following properties

1. all leaves have the same distance to the root

2. every internal non-root vertex v has at least a and at most b children

3. the root has degree at least 2 if the tree is non-empty

4. the internal vertices do not contain data, but only keys (external search tree)

5. there is a special dummy leaf node with key-value ∞
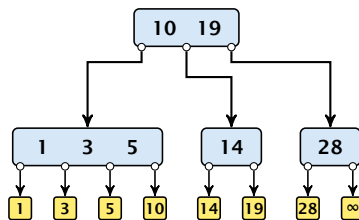
7.5 (a, b)-trees

Each internal node v with $d(v)$ children stores $d - 1$ keys $k_1, \ldots, k_{d-1}$. The i-th subtree of v fulfills

$$k_{i-1} < \text{ key in } i\text{-th sub-tree } \leq k_i ,$$

where we use $k_0 = -\infty$ and $k_d = \infty$.

7.5 (a, b)-trees

Example 2

7.5 (a, b)-trees

Variants

- The dummy leaf element may not exist; it only makes implementation more convenient.

- Variants in which $b = 2a$ are commonly referred to as B-trees.

- A B-tree usually refers to the variant in which keys and data are stored at internal nodes.

- A B^+ tree stores the data only at leaf nodes as in our definition. Sometimes the leaf nodes are also connected in a linear list data structure to speed up the computation of successors and predecessors.

- A B^* tree requires that a node is at least $2/3$-full as opposed to $1/2$-full (the requirement of a B-tree).

Lemma 3

Let T be an (a, b)-tree for $n > 0$ elements (i.e., $n + 1$ leaf nodes) and height h (number of edges from root to a leaf vertex). Then

   1. $2a^{h-1} \le n + 1 \le b^h$

   2. $\log_b(n + 1) \le h \le 1 + \log_a(\frac{n+1}{2})$
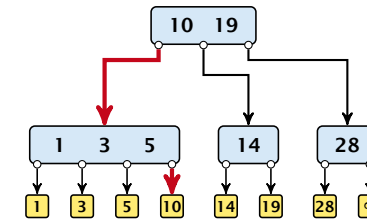
Proof.

- ▶ If $n > 0$ the root has degree at least 2 and all other nodes have degree at least a. This gives that the number of leaf nodes is at least $2a^{h-1}$.

- ▶ Analogously, the degree of any node is at most b and, hence, the number of leaf nodes at most b^h.
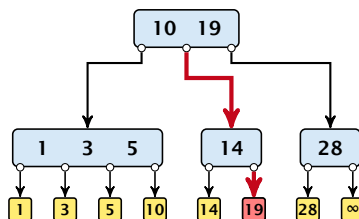
$\square$

Search

Search(8)

The search is straightforward. It is only important that you need to go all the way to the leaf.

Time: $\mathcal{O}(b \cdot h) = \mathcal{O}(b \cdot \log n)$, if the individual nodes are organized as linear lists.

Search

Search(19)

The search is straightforward. It is only important that you need to go all the way to the leaf.

Time: $\mathcal{O}(b \cdot h) = \mathcal{O}(b \cdot \log n)$, if the individual nodes are organized as linear lists.

Insert

Insert element x:

- ▶ Follow the path as if searching for $\text{key}[x]$.

- ▶ If this search ends in leaf ℓ, insert x before this leaf.

- ▶ For this add $\text{key}[x]$ to the key-list of the last internal node v on the path.

- ▶ If after the insert v contains b nodes, do Rebalance(v).

Insert

Rebalance(v):

- ▶ Let k_i, $i = 1, \ldots, b$ denote the keys stored in v.
- ▶ Let $j := \lfloor \frac{b+1}{2} \rfloor$ be the middle element.
- ▶ Create two nodes v_1, and v_2. v_1 gets all keys $k_1, \ldots, k_{j-1}$ and v_2 gets keys $k_{j+1}, \ldots, k_b$.
- ▶ Both nodes get at least $\lfloor \frac{b-1}{2} \rfloor$ keys, and have therefore degree at least $\lfloor \frac{b-1}{2} \rfloor + 1 \geq a$ since $b \geq 2a - 1$.
- ▶ They get at most $\lceil \frac{b-1}{2} \rceil$ keys, and have therefore degree at most $\lceil \frac{b-1}{2} \rceil + 1 \leq b$ (since $b \geq 2$).
- ▶ The key k_j is promoted to the parent of v. The current pointer to v is altered to point to v_1, and a new pointer (to the right of k_j) in the parent is added to point to v_2.
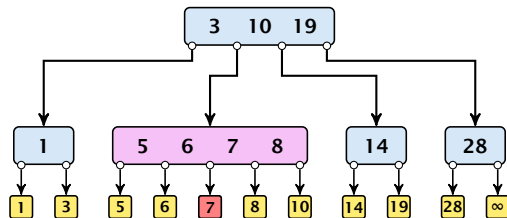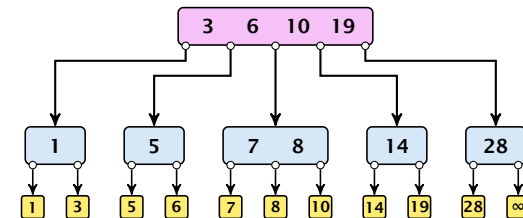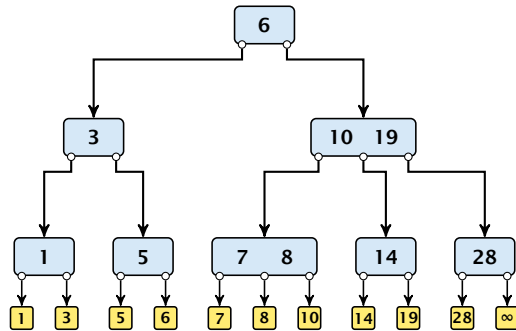- ▶ Then, re-balance the parent.

Insert

Insert(7)
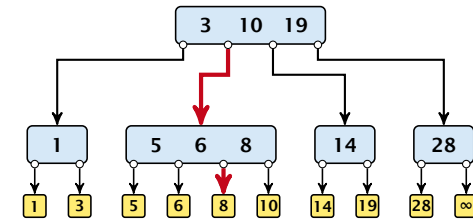
Insert

Insert(7)

Insert

Insert(7)

Insert

Insert(7)

Delete

Delete element x (pointer to leaf vertex):

▸ Let v denote the parent of x. If $\text{key}[x]$ is contained in v, remove the key from v, and delete the leaf vertex.

▸ Otherwise delete the key of the predecessor of x from v; delete the leaf vertex; and replace the occurrence of $\text{key}[x]$ in internal nodes by the predecessor key. (Note that it appears in exactly one internal vertex).

▸ If now the number of keys in v is below $a - 1$ perform Rebalance$'(v)$.


Delete

Rebalance$'(v)$:

▸ If there is a neighbour of v that has at least a keys take over the largest (if right neighbor) or smallest (if left neighbour) and the corresponding sub-tree.

▸ If not: merge v with one of its neighbours.

▸ The merged node contains at most $(a - 2) + (a - 1) + 1$ keys, and has therefore at most $2a - 1 \leq b$ successors.

▸ Then rebalance the parent.

▸ During this process the root may become empty. In this case the root is deleted and the height of the tree decreases.

Delete

Animation for deleting in an (a, b)-tree is only available in the lecture version of the slides.
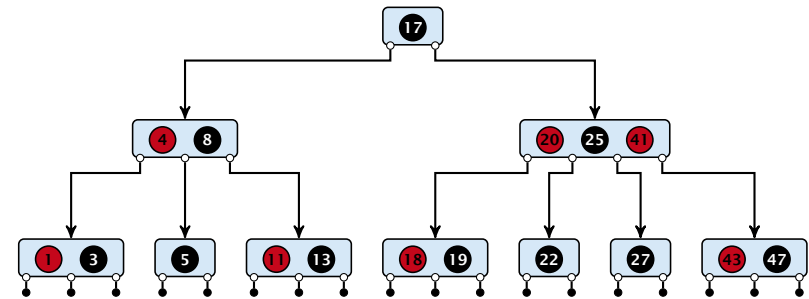
$(2,4)$-trees and red black trees

There is a close relation between red-black trees and $(2,4)$-trees:
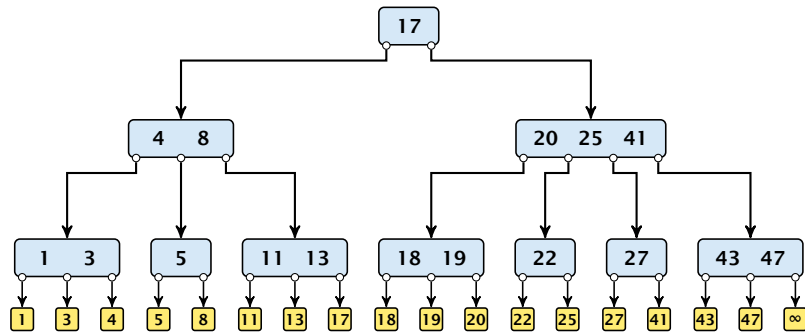
First make it into an internal search tree by moving the satellite-data from the leaves to internal nodes. Add dummy leaves.

7.5 (a,b)-trees

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$(2,4)$-trees and red black trees

There is a close relation between red-black trees and $(2,4)$-trees:

Then, color one key in each internal node v black. If v contains 3 keys you need to select the middle key otherwise choose a black key arbitrarily. The other keys are colored red.

7.5 (a,b)-trees

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$(2,4)$-trees and red black trees

There is a close relation between red-black trees and $(2,4)$-trees:
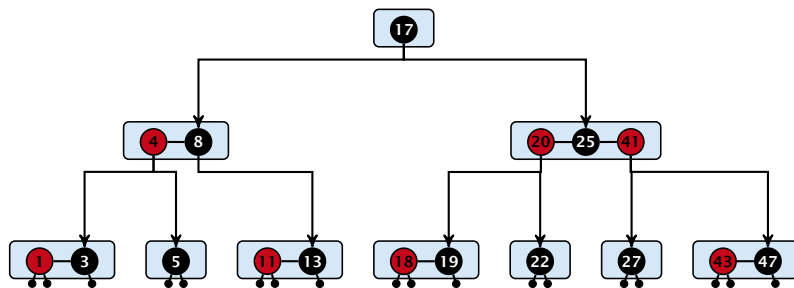
Re-attach the pointers to individual keys. A pointer that is between two keys is attached as a child of the red key. The incoming pointer, points to the black key.

7.5 (a,b)-trees

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$(2,4)$-trees and red black trees

There is a close relation between red-black trees and $(2,4)$-trees:
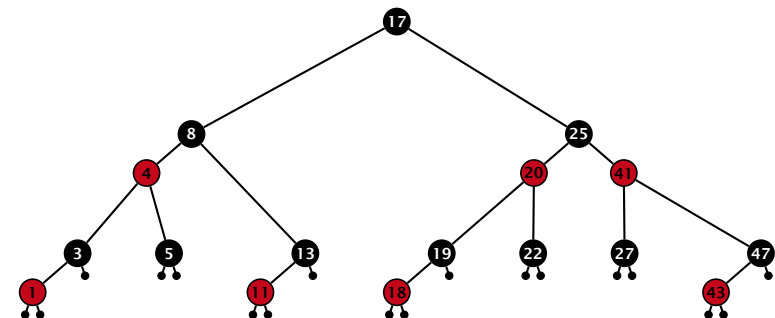
Note that this correspondence is not unique. In particular, there are different red-black trees that correspond to the same $(2,4)$-tree.

7.5 (a,b)-trees

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Augmenting Data Structures

Bibliography

[MS08]   Kurt Mehlhorn, Peter Sanders:
Algorithms and Data Structures — The Basic Toolbox,
Springer, 2008

[CLRS90] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein:
Introduction to algorithms (3rd ed.),
MIT Press and McGraw-Hill, 2009

A description of B-trees (a specific variant of (a, b)-trees) can be found in Chapter 18 of [CLRS90].
Chapter 7.2 of [MS08] discusses (a, b)-trees as discussed in the lecture.