6.5 Transformation of the Recurrence

Example 6

$$
\begin{aligned}
f_0 &= 1 \\
f_1 &= 2 \\
f_n &= f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 \,.
\end{aligned}
$$

Define

$$g_n := \log f_n \,.$$

Then

$$
\begin{aligned}
g_n &= g_{n-1} + g_{n-2} \text{ for } n \geq 2 \\
g_1 &= \log 2 = 1 (\text{for } \log = \log_2),\ g_0 = 0 \\
g_n &= F_n \ (n\text{-th Fibonacci number}) \\
f_n &= 2^{F_n}
\end{aligned}
$$

6.5 Transformation of the Recurrence

Example 7

$$
\begin{aligned}
f_1 &= 1 \\
f_n &= 3f_{\frac{n}{2}} + n; \text{ for } n = 2^k,\ k \geq 1\ ;
\end{aligned}
$$

Define

$$g_k := f_{2^k}\ .$$

Then:

$$
\begin{aligned}
g_0 &= 1 \\
g_k &= 3g_{k-1} + 2^k,\ k \geq 1
\end{aligned}
$$

6 Recurrences

We get

$$
\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
&= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1}
\end{aligned}
$$

6 Recurrences

Let $n = 2^k$:

$$\begin{aligned} g_k &= 3^{k+1} - 2^{k+1}, \text{ hence} \\ f_n &= 3 \cdot 3^k - 2 \cdot 2^k \\ &= 3(2^{\log 3})^k - 2 \cdot 2^k \\ &= 3(2^k)^{\log 3} - 2 \cdot 2^k \\ &= 3n^{\log 3} - 2n \ . \end{aligned}$$

6 Recurrences

Bibliography

[MS08] Kurt Mehlhorn, Peter Sanders:
Algorithms and Data Structures — The Basic Toolbox,
Springer, 2008

[CLRS90] Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein:
Introduction to algorithms (3rd ed.),
MIT Press and McGraw-Hill, 2009

[Liu85] Chung Laung Liu:
Elements of Discrete Mathematics
McGraw-Hill, 1985

The Karatsuba method can be found in [MS08] Chapter 1. Chapter 4.3 of [CLRS90] covers the "Substitution method" which roughly corresponds to "Guessing+induction". Chapters 4.4, 4.5, 4.6 of this book cover the master theorem. Methods using the characteristic polynomial and generating functions can be found in [Liu85] Chapter 10.