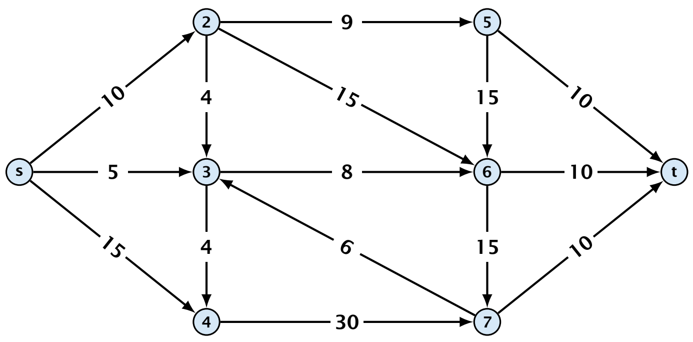
11 Introduction

Flow Network

- directed graph $G = (V, E)$; edge capacities $c(e)$
- two special nodes: source s; target t;
- no edges entering s or leaving t;
- at least for now: no parallel edges;



Cuts

Definition 1
An (s, t)-cut in the graph G is given by a set $A \subset V$ with $s \in A$ and $t \in V \setminus A$.

Definition 2
The capacity of a cut A is defined as

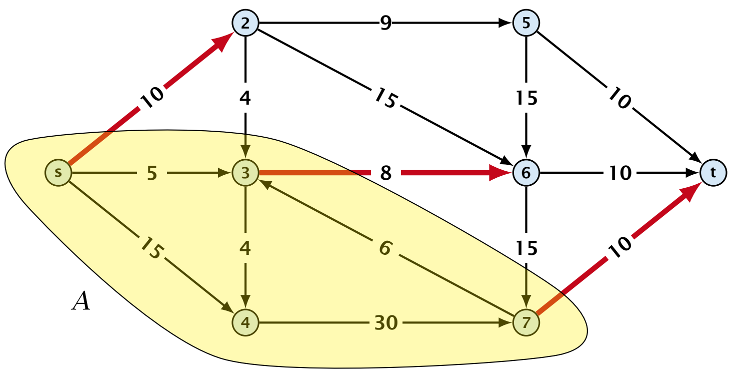
$$\mathrm{cap}(A, V \setminus A) := \sum_{e \in \mathrm{out}(A)} c(e) \ ,$$

where $\mathrm{out}(A)$ denotes the set of edges of the form $A \times V \setminus A$ (i.e. edges leaving A).

Minimum Cut Problem: Find an (s, t)-cut with minimum capacity.

Cuts

Example 3



The capacity of the cut is $\mathrm{cap}(A, V \setminus A) = 28$.

Flows

Definition 4
An (s, t)-flow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e
$$0 \le f(e) \le c(e) \ .$$
(capacity constraints)
2. For each $v \in V \setminus \{s, t\}$
$$\sum_{e \in \mathrm{out}(v)} f(e) = \sum_{e \in \mathrm{into}(v)} f(e) \ .$$
(flow conservation constraints)

Flows

Definition 5

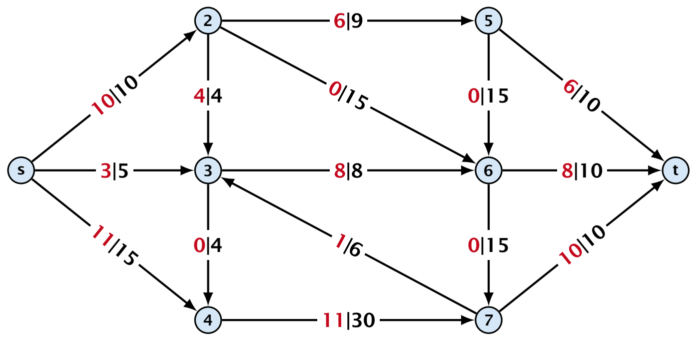
The value of an (s,t)-flow f is defined as

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(s)} f(e) \ .$$

Maximum Flow Problem: Find an (s,t)-flow with maximum value.

Flows

Example 6



The value of the flow is $\mathrm{val}(f) = 24$.

Flows

Lemma 7 (Flow value lemma)

Let f be a flow, and let $A \subseteq V$ be an (s,t)-cut. Then the net-flow across the cut is equal to the amount of flow leaving s, i.e.,

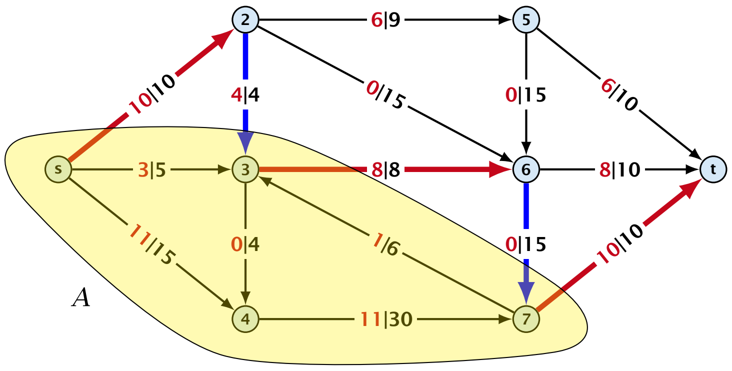
$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(A)} f(e) - \sum_{e \in \mathrm{into}(A)} f(e) \ .$$

Proof.

$$\begin{aligned}
\mathrm{val}(f) &= \sum_{e \in \mathrm{out}(s)} f(e) \\
&= \sum_{e \in \mathrm{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \Bigg(\sum_{e \in \mathrm{out}(v)} f(e) - \sum_{e \in \mathrm{in}(v)} f(e) \Bigg) \\
&= \sum_{e \in \mathrm{out}(A)} f(e) - \sum_{e \in \mathrm{into}(A)} f(e)
\end{aligned}$$

The last equality holds since every edge with both end-points in A contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering A. □

Example 8



Corollary 9

Let f be an (s,t)-flow and let A be an (s,t)-cut, such that

$$\mathrm{val}(f) = \mathrm{cap}(A, V \setminus A).$$

Then f is a maximum flow.

Proof.

Suppose that there is a flow f' with larger value. Then

$$\begin{aligned}
\mathrm{cap}(A, V \setminus A) &< \mathrm{val}(f') \\
&= \sum_{e \in \mathrm{out}(A)} f'(e) - \sum_{e \in \mathrm{into}(A)} f'(e) \\
&\leq \sum_{e \in \mathrm{out}(A)} f'(e) \\
&\leq \mathrm{cap}(A, V \setminus A)
\end{aligned}$$

□