# Bin Packing

Given $n$ items with sizes $s_1, \ldots, s_n$ where

$$1 > s_1 \geq \cdots \geq s_n > 0 \ .$$

Pack items into a minimum number of bins where each bin can hold items of total size at most 1.

**Theorem 5**

*There is no $\rho$-approximation for Bin Packing with $\rho < 3/2$ unless* $\mathrm{P} = \mathrm{NP}$.

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**Proof**

- In the partition problem we are given positive integers $b_1, \ldots, b_n$ with $B = \sum_i b_i$ even. Can we partition the integers into two sets $S$ and $T$ s.t.

$$\sum_{i \in S} b_i = \sum_{i \in T} b_i \quad ?$$

- We can solve this problem by setting $s_i := 2b_i/B$ and asking whether we can pack the resulting items into 2 bins or not.
- A $\rho$-approximation algorithm with $\rho < 3/2$ cannot output 3 or more bins when 2 are optimal.
- Hence, such an algorithm can solve Partition.

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### Definition 6

An asymptotic polynomial-time approximation scheme (APTAS) is a family of algorithms $\{A_\epsilon\}$ along with a constant $c$ such that $A_\epsilon$ returns a solution of value at most $(1+\epsilon)\mathrm{OPT}+c$ for minimization problems.

Note that for Set Cover or for Knapsack it makes no sense to differentiate between the notion of a PTAS or an APTAS because of scaling.

However, we will develop an APTAS for Bin Packing.

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# Bin Packing

Again we can differentiate between small and large items.

## Lemma 7

*Any packing of items into $\ell$ bins can be extended with items of size at most $\gamma$ s.t. we use only $\max\{\ell, \frac{1}{1-\gamma}\mathrm{SIZE}(I) + 1\}$ bins, where $\mathrm{SIZE}(I) = \sum_i s_i$ is the sum of all item sizes.*

- If after Greedy we use more than $\ell$ bins, all bins (apart from the last) must be full to at least $1 - \gamma$.
- Hence, $r(1 - \gamma) \leq \mathrm{SIZE}(I)$ where $r$ is the number of nearly-full bins.
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Choose $\gamma = \epsilon/2$. Then we either use $\ell$ bins or at most

$$\frac{1}{1-\epsilon/2} \cdot \mathrm{OPT} + 1 \leq (1+\epsilon) \cdot \mathrm{OPT} + 1$$

bins.

It remains to find an algorithm for the large items.

# Bin Packing

**Linear Grouping:**

Generate an instance $I'$ (for large items) as follows.

- Order large items according to size.
- Let the first $k$ items belong to group 1; the following $k$ items belong to group 2; etc.
- Delete items in the first group;
- Round items in the remaining groups to the size of the largest item in the group.

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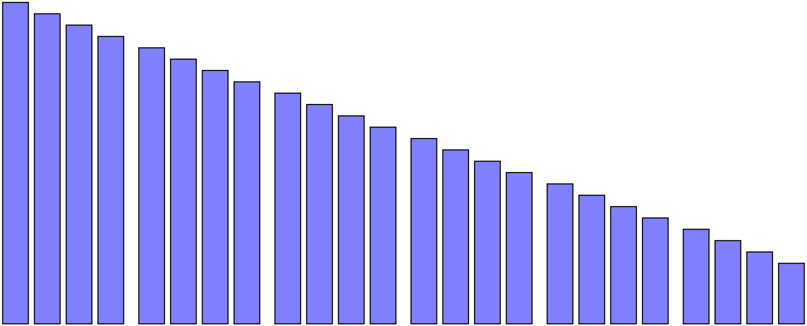

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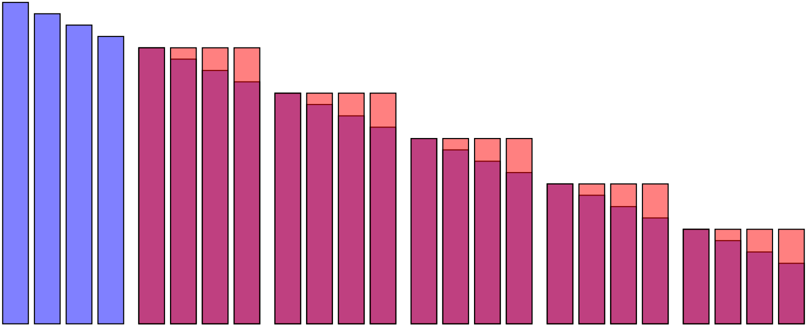

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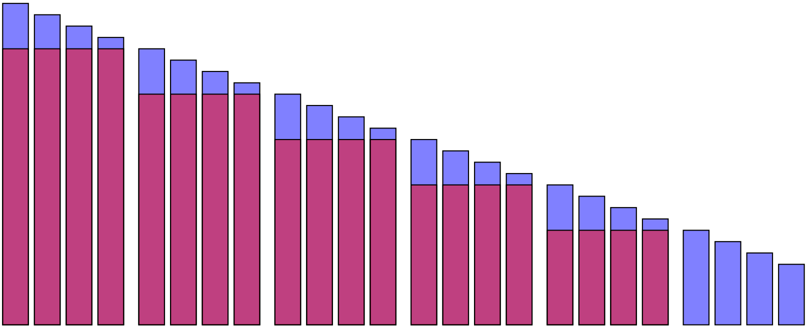

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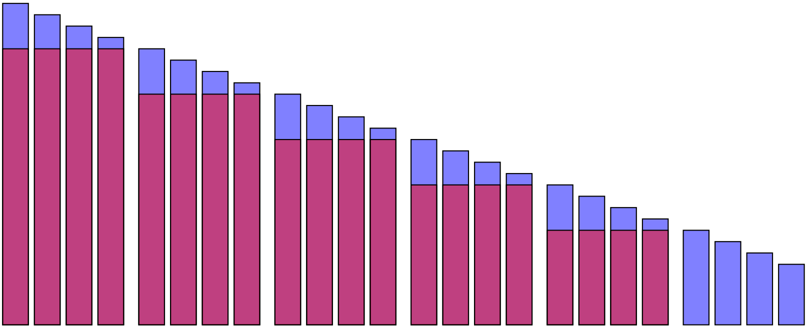

## Lemma 8

$\mathrm{OPT}(I') \le \mathrm{OPT}(I) \le \mathrm{OPT}(I') + k$

**Proof 1:**

- Any bin packing for $I$ gives a bin packing for $I'$ as follows.
- Pack the items of group 2, where in the packing for $I$ the items for group 1 have been packed;
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**Proof 2:**

- Any bin packing for $I'$ gives a bin packing for $I$ as follows.
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Assume that our instance does not contain pieces smaller than $\epsilon/2$. Then $\mathrm{SIZE}(I) \geq \epsilon n/2$.

We set $k = \lfloor \epsilon \mathrm{SIZE}(I) \rfloor$.

Then $n/k \leq n/\lfloor \epsilon^2 n/2 \rfloor \leq 4/\epsilon^2$ (here we used $\lfloor \alpha \rfloor \geq \alpha/2$ for $\alpha \geq 1$).

Hence, after grouping we have a constant number of piece sizes ($4/\epsilon^2$) and at most a constant number ($2/\epsilon$) can fit into any bin.

We can find an optimal packing for such instances by the previous Dynamic Programming approach.

- cost (for large items) at most

$$\mathrm{OPT}(I') + k \leq \mathrm{OPT}(I) + \epsilon \mathrm{SIZE}(I) \leq (1+\epsilon)\mathrm{OPT}(I)$$

- running time $\mathcal{O}((\frac{2}{\epsilon} n)^{4/\epsilon^2})$.

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