

Problem definition:

- ▶ n Boolean variables
- ▶ m clauses C_1, \dots, C_m . For example

$$C_7 = x_3 \vee \bar{x}_5 \vee \bar{x}_9$$

- ▶ Non-negative weight w_j for each clause C_j .
- ▶ Find an assignment of true/false to the variables such that the total weight of clauses that are **satisfied** is maximum.

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Terminology:

- ▶ A variable x_i and its negation \bar{x}_i are called **literals**.
- ▶ Hence, each clause consists of a set of literals (i.e., no duplications: $x_i \vee x_i \vee \bar{x}_j$ is not a clause).
- ▶ We assume a clause does not contain x_i and \bar{x}_i for any i .
- ▶ x_i is called a **positive literal** while the negation \bar{x}_i is called a **negative literal**.
- ▶ For a given clause C_j the number of its literals is called its **length** or **size** and denoted with l_j .
- ▶ Clauses of length one are called **unit clauses**.

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MAXSAT: Flipping Coins

Set each x_i independently to **true** with probability $\frac{1}{2}$ (and, hence, to **false** with probability $\frac{1}{2}$, as well).

Define random variable X_j with

$$X_j = \begin{cases} 1 & \text{if } C_j \text{ satisfied} \\ 0 & \text{otw.} \end{cases}$$

Then the total weight W of satisfied clauses is given by

$$W = \sum_j w_j X_j$$

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$$E[W]$$

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MAXSAT: LP formulation

- ▶ Let for a clause C_j , P_j be the set of positive literals and N_j the set of negative literals.

$$C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$$

$$\begin{array}{ll} \max & \sum_j w_j z_j \\ \text{s.t.} & \forall j \quad \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j \\ & \forall i \quad y_i \in \{0, 1\} \\ & \forall j \quad z_j \leq 1 \end{array}$$

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MAXSAT: Randomized Rounding

Set each x_i independently to **true** with probability y_i (and, hence, to **false** with probability $(1 - y_i)$).

Lemma 7 (Geometric Mean \leq Arithmetic Mean)

For any nonnegative a_1, \dots, a_k

$$\left(\prod_{i=1}^k a_i \right)^{1/k} \leq \frac{1}{k} \sum_{i=1}^k a_i$$

Definition 8

A function f on an interval I is **concave** if for any two points s and r from I and any $\lambda \in [0, 1]$ we have

$$f(\lambda s + (1 - \lambda)r) \geq \lambda f(s) + (1 - \lambda)f(r)$$

Lemma 9

Let f be a concave function on the interval $[0, 1]$, with $f(0) = a$ and $f(1) = a + b$. Then

$$\begin{aligned} f(\lambda) &= f((1 - \lambda)0 + \lambda 1) \\ &\geq (1 - \lambda)f(0) + \lambda f(1) \\ &= a + \lambda b \end{aligned}$$

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$\Pr[C_j \text{ not satisfied}]$

$$\Pr[C_j \text{ not satisfied}] = \prod_{i \in P_j} (1 - y_i) \prod_{i \in N_j} y_i$$

$$\begin{aligned}\Pr[C_j \text{ not satisfied}] &= \prod_{i \in P_j} (1 - y_i) \prod_{i \in N_j} y_i \\ &\leq \left[\frac{1}{\ell_j} \left(\sum_{i \in P_j} (1 - y_i) + \sum_{i \in N_j} y_i \right) \right]^{\ell_j}\end{aligned}$$

$$\begin{aligned}
\Pr[C_j \text{ not satisfied}] &= \prod_{i \in P_j} (1 - y_i) \prod_{i \in N_j} y_i \\
&\leq \left[\frac{1}{\ell_j} \left(\sum_{i \in P_j} (1 - y_i) + \sum_{i \in N_j} y_i \right) \right]^{\ell_j} \\
&= \left[1 - \frac{1}{\ell_j} \left(\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \right) \right]^{\ell_j}
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&\leq \left(1 - \frac{z_j}{\ell_j} \right)^{\ell_j} .
\end{aligned}$$

The function $f(z) = 1 - (1 - \frac{z}{\ell})^\ell$ is concave. Hence,

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$f''(z) = -\frac{\ell-1}{\ell} \left[1 - \frac{z}{\ell}\right]^{\ell-2} \leq 0$ for $z \in [0, 1]$. Therefore, f is concave.

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MAXSAT: The better of two

Theorem 10

Choosing the better of the two solutions given by randomized rounding and coin flipping yields a $\frac{3}{4}$ -approximation.

Let W_1 be the value of randomized rounding and W_2 the value obtained by coin flipping.

$$E[\max\{W_1, W_2\}]$$

Let W_1 be the value of randomized rounding and W_2 the value obtained by coin flipping.

$$\begin{aligned} E[\max\{W_1, W_2\}] \\ \geq E[\frac{1}{2}W_1 + \frac{1}{2}W_2] \end{aligned}$$

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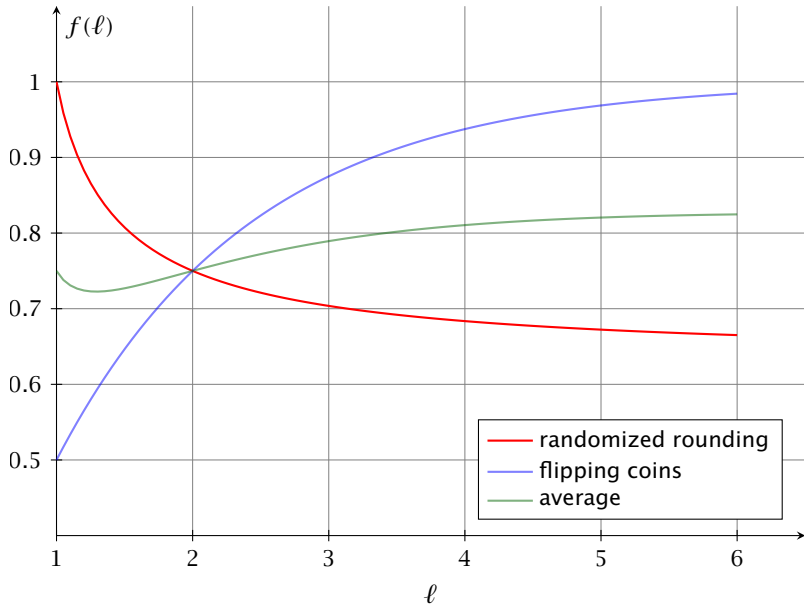
$$\begin{aligned} E[\max\{W_1, W_2\}] &\geq E[\frac{1}{2}W_1 + \frac{1}{2}W_2] \\ &\geq \frac{1}{2} \sum_j w_j z_j \left[1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j} \right] + \frac{1}{2} \sum_j w_j \left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right) \end{aligned}$$

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 & \geq \frac{3}{4} \text{OPT}
 \end{aligned}$$



MAXSAT: Nonlinear Randomized Rounding

So far we used **linear** randomized rounding, i.e., the probability that a variable is set to 1/true was exactly the value of the corresponding variable in the linear program.

We could define a function $f : [0, 1] \rightarrow [0, 1]$ and set x_i to true with probability $f(y_i)$.

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MAXSAT: Nonlinear Randomized Rounding

Let $f : [0, 1] \rightarrow [0, 1]$ be a function with

$$1 - 4^{-x} \leq f(x) \leq 4^{x-1}$$

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Rounding the LP-solution with a function f of the above form gives a $\frac{3}{4}$ -approximation.

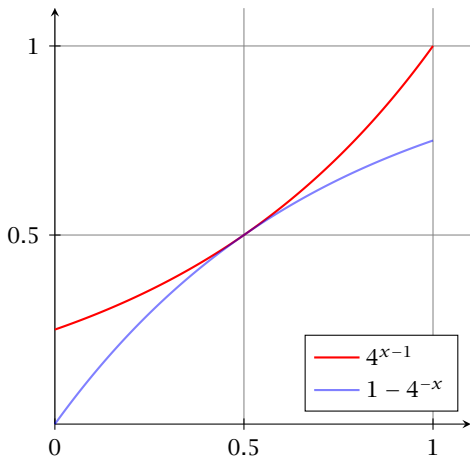
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Can we do better?

Not if we compare ourselves to the value of an optimum LP-solution.

Definition 12 (Integrality Gap)

The integrality gap for an ILP is the worst-case ratio over all instances of the problem of the value of an optimal IP-solution to the value of an optimal solution to its linear programming relaxation.

Note that the integrality is less than one for maximization problems and larger than one for minimization problems (of course, equality is possible).

Note that an integrality gap only holds for one specific ILP formulation.

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Lemma 13

Our ILP-formulation for the MAXSAT problem has integrality gap at most $\frac{3}{4}$.

$$\begin{array}{ll} \max & \sum_j w_j z_j \\ \text{s.t.} & \forall j \quad \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j \\ & \forall i \quad y_i \in \{0, 1\} \\ & \forall j \quad z_j \leq 1 \end{array}$$

Consider: $(x_1 \vee x_2) \wedge (\bar{x}_1 \vee x_2) \wedge (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_2)$

- ▶ any solution can satisfy at most 3 clauses
- ▶ we can set $y_1 = y_2 = 1/2$ in the LP; this allows to set $z_1 = z_2 = z_3 = z_4 = 1$
- ▶ hence, the LP has value 4.

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$$\begin{array}{ll} \max & \sum_j w_j z_j \\ \text{s.t.} & \forall j \quad \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j \\ & \forall i \quad y_i \in \{0, 1\} \\ & \forall j \quad z_j \leq 1 \end{array}$$

Consider: $(x_1 \vee x_2) \wedge (\bar{x}_1 \vee x_2) \wedge (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_2)$

- ▶ any solution can satisfy at most 3 clauses
- ▶ we can set $y_1 = y_2 = 1/2$ in the LP; this allows to set $z_1 = z_2 = z_3 = z_4 = 1$
- ▶ hence, the LP has value 4.