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We first solve the LP-relaxation and then we round the fractional values so that we obtain an integral solution.

Set Cover relaxation:

$$\begin{array}{llrcl} \min & & \sum_{i=1}^{k} w_i x_i & & \\ \text{s.t.} & \forall u \in U & \sum_{i:u\in S_i} x_i & \geq & 1 \\ & \forall i \in \{1,\dots,k\} & x_i & \in & [0,1] \end{array}$$

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Rounding Algorithm:

Set all x_i-values with $x_i \geq \frac{1}{f}$ to 1. Set all other x_i-values to 0.

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Lemma 2
The rounding algorithm gives an f-approximation.

Proof: Every $u \in U$ is covered.

- We know that $\sum_{i: u \in S_i} x_i \geq 1$.
- The sum contains at most $f_u \leq f$ elements.
- Therefore one of the sets that contain u must have $x_i \geq 1/f$.
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Technique 2: Rounding the Dual Solution.

Relaxation for Set Cover

Primal:

$$\begin{array}{lrl} \min & \multicolumn{2}{l}{\sum_{i\in I} w_i x_i} \\ \text{s.t. } \forall u & \sum_{i:u\in S_i} x_i & \geq 1 \\ & x_i & \geq 0 \end{array}$$

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Technique 2: Rounding the Dual Solution.

Rounding Algorithm:

Let I denote the index set of sets for which the dual constraint is tight. This means for all $i \in I$

$$\sum_{u:u\in S_i} y_u = w_i$$

Technique 2: Rounding the Dual Solution.

Lemma 3

The resulting index set is an f-approximation.

Proof:

Every $u \in U$ is covered.

- Suppose there is a u that is not covered.
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- But then y_u could be increased in the dual solution without violating any constraint. This is a contradiction to the fact that the dual solution is optimal.

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Let I denote the solution obtained by the first rounding algorithm and I' be the solution returned by the second algorithm. Then

$$I \subseteq I' \ .$$

This means I' is never better than I.

Suppose that we take S_i in the first algorithm. I.e., $i \in I$.

This means $x_i^* \geq \frac{1}{f}$.

Because of Complementary Slackness Conditions the corresponding constraint in the dual must be tight.

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Technique 3: The Primal Dual Method

The previous two rounding algorithms have the disadvantage that it is necessary to solve the LP. The following method also gives an f-approximation without solving the LP.

For estimating the cost of the solution we only required two properties.

1. The solution is dual feasible and, hence,

$$\sum_u y_u \le \mathrm{cost}(x^*) \le \mathrm{OPT}$$

   where x^* is an optimum solution to the primal LP.

2. The set I contains only sets for which the dual inequality is tight.

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Technique 3: The Primal Dual Method

Algorithm 1 PrimalDual

$y \leftarrow 0$
$I \leftarrow \emptyset$
while exists $u \notin \bigcup_{i \in I} S_i$ **do**
    increase dual variable y_u until constraint for some new set S_ℓ becomes tight
    $I \leftarrow I \cup \{\ell\}$

Technique 4: The Greedy Algorithm

Algorithm 1 Greedy

1: $I \leftarrow \emptyset$
2: $\hat{S}_j \leftarrow S_j$ for all j
3: **while** I not a set cover **do**
4: $\quad \ell \leftarrow \arg\min_{j:\hat{S}_j \neq 0} \frac{w_j}{|\hat{S}_j|}$
5: $\quad I \leftarrow I \cup \{\ell\}$
6: $\quad \hat{S}_j \leftarrow \hat{S}_j - S_\ell$ for all j

In every round the Greedy algorithm takes the set that covers remaining elements in the most cost-effective way.

We choose a set such that the ratio between cost and still uncovered elements in the set is minimized.

Technique 4: The Greedy Algorithm

Lemma 4

Given positive numbers $a_1, \dots, a_k$ *and* $b_1, \dots, b_k$*, and* $S \subseteq \{1, \dots, k\}$ *then*

$$\min_i \frac{a_i}{b_i} \le \frac{\sum_{i \in S} a_i}{\sum_{i \in S} b_i} \le \max_i \frac{a_i}{b_i}$$

Technique 4: The Greedy Algorithm

Let n_ℓ denote the number of elements that remain at the beginning of iteration ℓ. $n_1 = n = |U|$ and $n_{s+1} = 0$ if we need s iterations.

In the ℓ-th iteration

$$\min_j \frac{w_j}{|\hat{S}_j|} \le \frac{\sum_{j\in\mathrm{OPT}} w_j}{\sum_{j\in\mathrm{OPT}} |\hat{S}_j|} = \frac{\mathrm{OPT}}{\sum_{j\in\mathrm{OPT}} |\hat{S}_j|} \le \frac{\mathrm{OPT}}{n_\ell}$$

since an optimal algorithm can cover the remaining n_ℓ elements with cost OPT.

Let $\hat{S}_j$ be a subset that minimizes this ratio. Hence, $w_j/|\hat{S}_j| \le \frac{\mathrm{OPT}}{n_\ell}$.

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Adding this set to our solution means $n_{\ell+1} = n_\ell - |\hat{S}_j|$.

$$w_j \le \frac{|\hat{S}_j|\mathrm{OPT}}{n_\ell} = \frac{n_\ell - n_{\ell+1}}{n_\ell} \cdot \mathrm{OPT}$$

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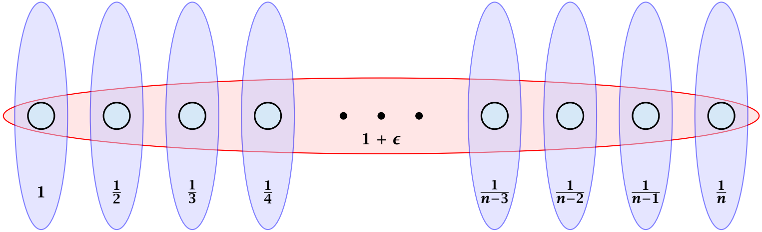
&= \mathrm{OPT} \sum_{i=1}^{k} \frac{1}{i} \\
&= H_n \cdot \mathrm{OPT} \leq \mathrm{OPT}(\ln n + 1) \enspace .
\end{aligned}
$$

Technique 4: The Greedy Algorithm

A tight example:



Technique 5: Randomized Rounding

One round of randomized rounding:
Pick set S_j uniformly at random with probability $1 - x_j$ (for all j).

Version A: Repeat rounds until you have a cover.

Version B: Repeat for s rounds. If you have a cover STOP. Otherwise, repeat the whole algorithm.

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Probability that $u \in U$ is not covered (in one round):

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Probability that $u \in U$ is not covered (after ℓ rounds):

$$
\Pr[u \text{ not covered after } \ell \text{ round}] \le \frac{1}{e^{\ell}} \enspace .
$$

$\Pr[\exists u \in U$ not covered after ℓ round$]$

$$
\begin{aligned}
&\Pr[\exists u \in U \text{ not covered after } \ell \text{ round}] \\
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Lemma 5

With high probability $\mathcal{O}(\log n)$ rounds suffice.

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With high probability:

For any constant α the number of rounds is at most $\mathcal{O}(\log n)$ with probability at least $1 - n^{-\alpha}$.

Proof: We have

$$\Pr[\#\text{rounds} \geq (\alpha+1)\ln n] \leq ne^{-(\alpha+1)\ln n} = n^{-\alpha} \ .$$

Expected Cost

- Version A.
  Repeat for $s = (\alpha + 1)\ln n$ rounds. If you don't have a cover simply take for each element u the cheapest set that contains u.

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for $n \ge 2$ and $\alpha \ge 1$.

Randomized rounding gives an $\mathcal{O}(\log n)$ approximation. The running time is polynomial with high probability.

Theorem 6 (without proof)

There is no approximation algorithm for set cover with approximation guarantee better than $\frac{1}{2}\log n$ unless NP has quasi-polynomial time algorithms (algorithms with running time $2^{\mathrm{poly}(\log n)}$).

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Integrality Gap

The integrality gap of the SetCover LP is $\Omega(\log n)$.

- $n = 2^k - 1$
- Elements are all vectors $\boldsymbol{i}$ over $GF[2]$ of length k (excluding zero vector).
- Every vector $\boldsymbol{j}$ defines a set as follows

$$S_{\boldsymbol{j}} := \{\boldsymbol{i} \mid \boldsymbol{i} \cdot \boldsymbol{j} = 1\}$$

- each set contains 2^{k-1} vectors; each vector is contained in 2^{k-1} sets
- $x_i = \frac{1}{2^{k-1}} = \frac{2}{n+1}$ is fractional solution.

Integrality Gap

Every collection of $p < k$ sets does not cover all elements.

Hence, we get a gap of $\Omega(\log n)$.

Techniques:

- Deterministic Rounding
- Rounding of the Dual
- Primal Dual
- Greedy
- Randomized Rounding
- Local Search
- Rounding Data + Dynamic Programming