Technique 4: The Greedy Algorithm

Algorithm 1 Greedy

1: $I \leftarrow \emptyset$
2: $\hat{S}_j \leftarrow S_j$ for all j
3: **while** I not a set cover **do**
4:      $\ell \leftarrow \arg\min_{j:\hat{S}_j \neq 0} \frac{w_j}{|\hat{S}_j|}$
5:      $I \leftarrow I \cup \{\ell\}$
6:      $\hat{S}_j \leftarrow \hat{S}_j - S_\ell$ for all j

In every round the Greedy algorithm takes the set that covers remaining elements in the most cost-effective way.

We choose a set such that the ratio between cost and still uncovered elements in the set is minimized.

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Lemma 4

Given positive numbers $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ then

$$\min_i \frac{a_i}{b_i} \leq \frac{\sum_i a_i}{\sum_i b_i} \leq \max_i \frac{a_i}{b_i}$$

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Let n_ℓ denote the number of elements that remain at the beginning of iteration ℓ. $n_1 = n = |U|$ and $n_{s+1} = 0$ if we need s iterations.

In the ℓ-th iteration

$$\min_j \frac{w_j}{|\hat{S}_j|} \leq \frac{\sum_{j \in \text{OPT}} w_j}{\sum_{j \in \text{OPT}} |\hat{S}_j|} = \frac{\text{OPT}}{\sum_{j \in \text{OPT}} |\hat{S}_j|} \leq \frac{\text{OPT}}{n_\ell}$$

since an optimal algorithm can cover the remaining n_ℓ elements with cost OPT.

Let $\hat{S}_j$ be a subset that minimizes this ratio. Hence, $w_j/|\hat{S}_j| \leq \frac{\text{OPT}}{n_\ell}$.

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Adding this set to our solution means $n_{\ell+1} = n_\ell - |\hat{S}_j|$.

$$w_j \le \frac{|\hat{S}_j|\text{OPT}}{n_\ell} = \frac{n_\ell - n_{\ell+1}}{n_\ell} \cdot \text{OPT}$$

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$$\sum_{j \in I} w_j \leq \sum_{\ell=1}^{s} \frac{n_\ell - n_{\ell+1}}{n_\ell} \cdot \mathrm{OPT}$$

$$\leq \mathrm{OPT} \sum_{\ell=1}^{s} \left(\frac{1}{n_\ell} + \frac{1}{n_\ell - 1} + \cdots + \frac{1}{n_{\ell+1} + 1} \right)$$

$$= \mathrm{OPT} \sum_{i=1}^{k} \frac{1}{i}$$

$$= H_n \cdot \mathrm{OPT} \leq \mathrm{OPT}(\ln n + 1) \ .$$