

Algorithm 1 FacilityLocation

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1:  $C \leftarrow D$  // unassigned clients
2:  $k \leftarrow 0$ 
3: while  $C \neq \emptyset$  do
4:    $k \leftarrow k + 1$ 
5:   choose  $j_k \in C$  that minimizes  $v_j^* + C_j^*$ 
6:   choose  $i_k \in N(j_k)$  according to probability  $x_{i,j_k}$ .
7:   assign  $j_k$  and all unassigned clients in  $N^2(j_k)$  to  $i_k$ 
8:    $C \leftarrow C - \{j_k\} - N^2(j_k)$ 
```

Total assignment cost:

- ▶ Fix k ; set $j = j_k$.
- ▶ Let $\ell \in N^2(j)$ and h (one of) its neighbour(s) in $N(j)$.
- ▶ If we assign a client ℓ to the same facility as i we pay at most

$$\sum_i c_{ij} x_{ij}^* + c_{hj} + c_{h\ell} \leq C_j^* + v_j^* + v_\ell^* \leq C_\ell^* + 2v_\ell^*$$

Summing this over all clients gives that the total assignment cost is at most

$$\sum_j C_j^* + \sum_j 2v_j^* \leq \sum_j C_j^* + 2\text{OPT}$$

Hence, it is at most 2OPT plus the total assignment cost in an optimum solution.

Adding the facility cost gives a 3-approximation.

Lemma 12 (Chernoff Bounds)

Let X_1, \dots, X_n be n independent 0-1 random variables, not necessarily identically distributed. Then for $X = \sum_{i=1}^n X_i$ and $\mu = E[X]$, $L \leq \mu \leq U$, and $\delta > 0$

$$\Pr[X \geq (1 + \delta)U] < \left(\frac{e^\delta}{(1 + \delta)^{1+\delta}} \right)^U,$$

and

$$\Pr[X \leq (1 - \delta)L] < \left(\frac{e^{-\delta}}{(1 - \delta)^{1-\delta}} \right)^L,$$

Lemma 13

For $0 \leq \delta \leq 1$ we have that

$$\left(\frac{e^\delta}{(1 + \delta)^{1+\delta}} \right)^U \leq e^{-U\delta^2/3}$$

and

$$\left(\frac{e^{-\delta}}{(1 - \delta)^{1-\delta}} \right)^L \leq e^{-L\delta^2/2}$$

Integer Multicommodity Flows

- ▶ Given s_i - t_i pairs in a graph.
- ▶ Connect each pair by a paths such that not too many path use any given edge.

$$\begin{array}{ll} \min & W \\ \text{s.t.} & \forall i \quad \sum_{p \in \mathcal{P}_i} x_p = 1 \\ & \sum_{p: e \in p} x_p \leq W \\ & x_p \in \{0, 1\} \end{array}$$

Integer Multicommodity Flows

Randomized Rounding:

For each i choose one path from the set \mathcal{P}_i at random according to the probability distribution given by the Linear Programming Solution.

Theorem 14

If $W^* \geq c \ln n$ for some constant c , then with probability at least $n^{-c/3}$ the total number of paths using any edge is at most $W^* + \sqrt{cW^* \ln n}$.

Integer Multicommodity Flows

Let X_e^i be a random variable that indicates whether the path for s_i - t_i uses edge e .

Then the number of paths using edge e is $Y_e = \sum_i X_e^i$.

$$E[Y_e] = \sum_i \sum_{p \in \mathcal{P}_i: e \in p} x_p^* = \sum_{p: e \in p} x_p^* \leq W^*$$

Integer Multicommodity Flows

Choose $\delta = \sqrt{(c \ln n)/W^*}$.

Then

$$\Pr[Y_e \geq (1 + \delta)W^*] < e^{-W^*\delta^2/3} = \frac{1}{n^{c/3}}$$

Repetition: Primal Dual for Set Cover

Primal Relaxation:

$$\begin{array}{ll} \min & \sum_{i=1}^k w_i x_i \\ \text{s.t.} & \forall u \in U \quad \sum_{i:u \in S_i} x_i \geq 1 \\ & \forall i \in \{1, \dots, k\} \quad x_i \geq 0 \end{array}$$

Dual Formulation:

$$\begin{array}{ll} \max & \sum_{u \in U} \gamma_u \\ \text{s.t.} & \forall i \in \{1, \dots, k\} \quad \sum_{u:u \in S_i} \gamma_u \leq w_i \\ & \gamma_u \geq 0 \end{array}$$

Repetition: Primal Dual for Set Cover

Algorithm:

- ▶ Start with $\gamma = 0$ (feasible dual solution).
Start with $x = 0$ (integral primal solution that may be infeasible).
- ▶ While x not feasible
 - ▶ Identify an element e that is not covered in current primal integral solution.
 - ▶ Increase dual variable γ_e until a dual constraint becomes tight (maybe increase by 0!).
 - ▶ If this is the constraint for set S_j set $x_j = 1$ (add this set to your solution).

Repetition: Primal Dual for Set Cover

Analysis:

- ▶ For every set S_j with $x_j = 1$ we have

$$\sum_{e \in S_j} \gamma_e = w_j$$

- ▶ Hence our cost is

$$\sum_j w_j = \sum_j \sum_{e \in S_j} \gamma_e = \sum_e |\{j : e \in S_j\}| \cdot \gamma_e \leq f \cdot \sum_e \gamma_e \leq f \cdot \text{OPT}$$