

6.4 Generating Functions

Definition 4 (Generating Function)

Let $(a_n)_{n \geq 0}$ be a sequence. The corresponding

- ▶ **generating function (Erzeugendenfunktion)** is

$$F(z) := \sum_{n \geq 0} a_n z^n;$$

- ▶ exponential generating function (exponentielle Erzeugendenfunktion) is

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There are two different views:

A generating function is a **formal power series** (formale Potenzreihe).

Then the generating function is an **algebraic object**.

Let $f = \sum_{n \geq 0} a_n z^n$ and $g = \sum_{n \geq 0} b_n z^n$.

- ▶ **Equality:** f and g are equal if $a_n = b_n$ for all n .
- ▶ **Addition:** $f + g := \sum_{n \geq 0} (a_n + b_n) z^n$.
- ▶ **Multiplication:** $f \cdot g := \sum_{n \geq 0} c_n z^n$ with $c = \sum_{p=0}^n a_p b_{n-p}$.

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The arithmetic view:

We view a power series as a function $f : \mathbb{C} \rightarrow \mathbb{C}$.

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What does $\sum_{n \geq 0} z^n = \frac{1}{1-z}$ mean in the **algebraic view**?

It means that the power series $1 - z$ and the power series $\sum_{n \geq 0} z^n$ are invers, i.e.,

$$(1 - z) \cdot \left(\sum_{n \geq 0} z^n \right) = 1 .$$

This is well-defined.

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Hence, the generating function of the sequence $a_n = n + 1$ is $1/(1-z)^2$.

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Hence, the generating function of the sequence $a_n = (n+1)(n+2)$ is $\frac{2}{(1-z)^3}$.

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The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

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The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

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We know

$$\sum_{n \geq 0} y^n = \frac{1}{1-y}$$

Hence,

$$\sum_{n \geq 0} a^n z^n = \frac{1}{1-az}$$

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Example: $a_n = a_{n-1} + 1, a_0 = 1$

Suppose we have the recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ and $a_0 = 1$.

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Hence, $a_n = n + 1$.

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$\frac{1}{n!}$	e^z

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Solving Recursions with Generating Functions

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Techniques:

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Techniques:
 - ▶ partial fraction decomposition (**Partialbruchzerlegung**)

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3. Do further transformations so that the infinite sums on the right hand side can be replaced by $A(z)$.
4. Solving for $A(z)$ gives an equation of the form $A(z) = f(z)$, where hopefully $f(z)$ is a simple function.
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 - ▶ partial fraction decomposition (**Partialbruchzerlegung**)
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Solving Recursions with Generating Functions

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6. The coefficients of the resulting power series are the a_n .

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This leads to the following conditions:

$$A + B + C = 1$$

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which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.