A Fast Matching Algorithm

Algorithm 50 Bimatch-Hopcroft-Karp(G)

1: $M \leftarrow \emptyset$
2: **repeat**
3: let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be maximal set of
4: vertex-disjoint, shortest augmenting path w.r.t. M.
5: $M \leftarrow M \oplus (P_1 \cup \cdots \cup P_k)$
6: **until** $\mathcal{P} = \emptyset$
7: **return** M

We call one iteration of the repeat-loop a phase of the algorithm.

Analysis

Lemma 4

Given a matching M and a maximal matching M^* there exist $|M^*| - |M|$ vertex-disjoint augmenting path w.r.t. M.

Proof:

- Similar to the proof that a matching is optimal iff it does not contain an augmenting paths.
- Consider the graph $G = (V, M \oplus M^*)$, and mark edges in this graph blue if they are in M and red if they are in M^*.
- The connected components of G are cycles and paths.
- The graph contains $k \geq |M^*| - |M|$ more red edges than blue edges.
- Hence, there are at least k components that form a path starting and ending with a blue edge. These are augmenting paths w.r.t. M.

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- Let $P_1, \ldots, P_k$ be a maximal collection of vertex-disjoint, shortest augmenting paths w.r.t. M (let $\ell = |P_i|$).
- $M' \stackrel{\text{def}}{=} M \oplus (P_1 \cup \cdots \cup P_k) = M \oplus P_1 \oplus \cdots \oplus P_k$.
- Let P be an augmenting path in M'.

Lemma 5

The set $A \stackrel{\text{def}}{=} M \oplus (M' \oplus P) = (P_1 \cup \cdots \cup P_k) \oplus P$ contains at least $(k+1)\ell$ edges.

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Proof.

- The set describes exactly the symmetric difference between matchings M and $M' \oplus P$.
- Hence, the set contains at least $k+1$ vertex-disjoint augmenting paths w.r.t. M as $|M'| = |M| + k + 1$.
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Lemma 6

P is of length at least $\ell + 1$. This shows that the length of a shortest augmenting path increases between two phases of the Hopcroft-Karp algorithm.

Proof.

- If P does not intersect any of the $P_1, \ldots, P_k$, this follows from the maximality of the set $\{P_1, \ldots, P_k\}$.
- Otherwise, at least one edge from P coincides with an edge from paths $\{P_1, \ldots, P_k\}$.
- This edge is not contained in A.
- Hence, $|A| \le k\ell + |P| - 1$.
- The lower bound on $|A|$ gives $(k+1)\ell \le |A| \le k\ell + |P| - 1$, and hence $|P| \ge \ell + 1$.

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If the shortest augmenting path w.r.t. a matching M has ℓ edges then the cardinality of the maximum matching is of size at most $|M| + \frac{|V|}{\ell+1}$.

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The symmetric difference between M and M^* contains $|M^*| - |M|$ vertex-disjoint augmenting paths. Each of these paths contains at least $\ell + 1$ vertices. Hence, there can be at most $\frac{|V|}{\ell+1}$ of them.

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The Hopcroft-Karp algorithm requires at most $2\sqrt{|V|}$ phases.

Proof.

- After iteration $\lfloor\sqrt{|V|}\rfloor$ the length of a shortest augmenting path must be at least $\lfloor\sqrt{|V|}\rfloor + 1 \geq \sqrt{|V|}$.
- Hence, there can be at most $|V|/(\sqrt{|V|}+1) \leq \sqrt{|V|}$ additional augmentations.

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Lemma 8

One phase of the Hopcroft-Karp algorithm can be implemented in time $\mathcal{O}(m)$.

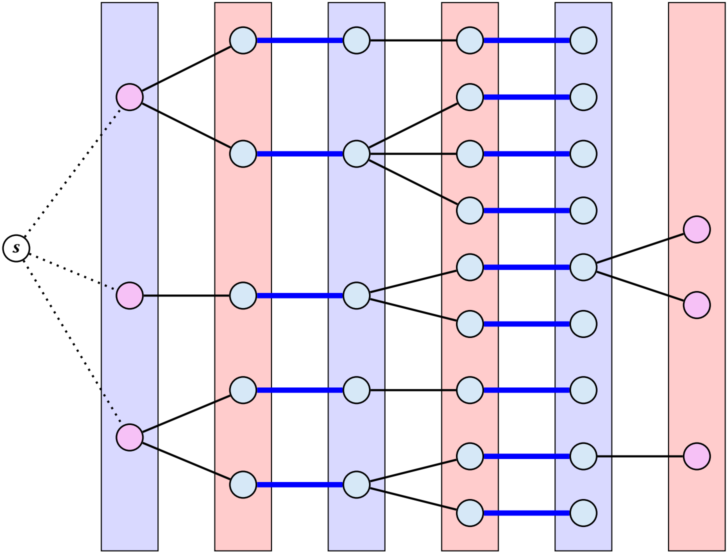
- Do a breadth first search starting at all free vertices in the left side L.

  (alternatively add a super-startnode; connect it to all free vertices in L and start breadth first search from there)

- The search stops when reaching a free vertex. However, the current level of the BFS tree is still finished in order to find a set F of free vertices (on the right side) that can be reached via shortest augmenting paths.

Analysis

- Then a maxim**al** set of shortest path from the leftmost layer of the tree construction to nodes in F needs to be computed.
- Any such path must visit the layers of the BFS-tree from left to right.
- To go from an odd layer to an even layer it must use a matching edge.
- To go from an even layer to an odd layer edge it can use edges in the BFS-tree **or** edges that have been ignored during BFS-tree construction.
- We direct all edges btw. an even node in some layer ℓ to an odd node in layer $\ell + 1$ from left to right.
- A DFS search in the resulting graph gives us a maximal set of vertex disjoint path from left to right in the resulting graph.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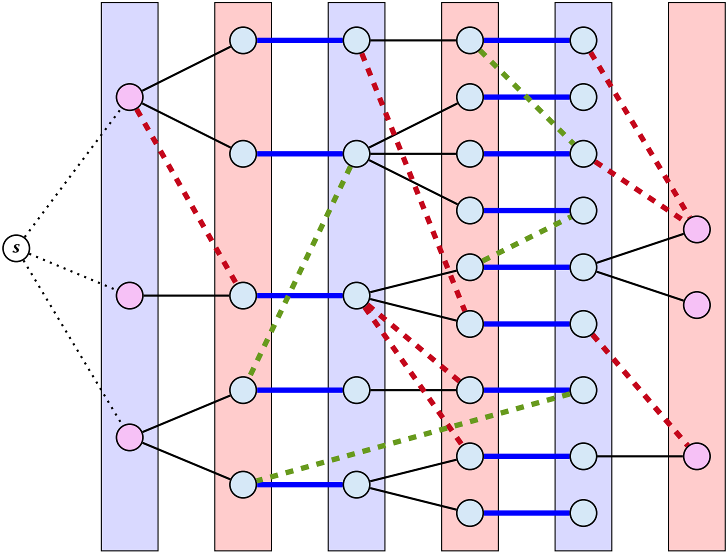
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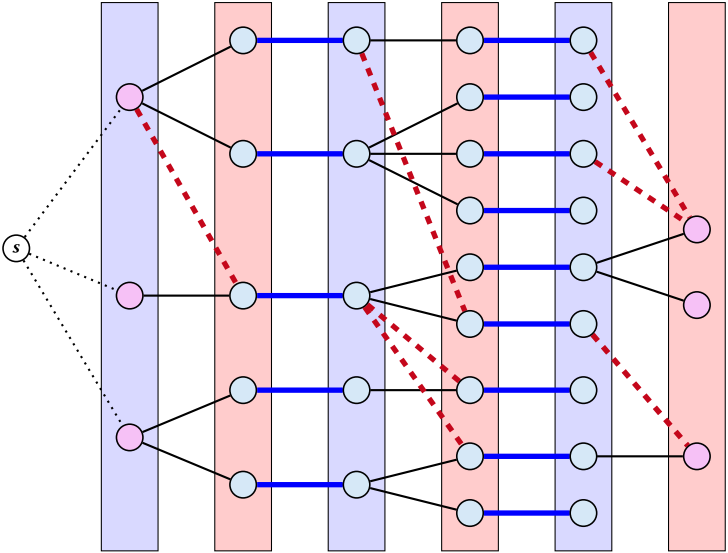
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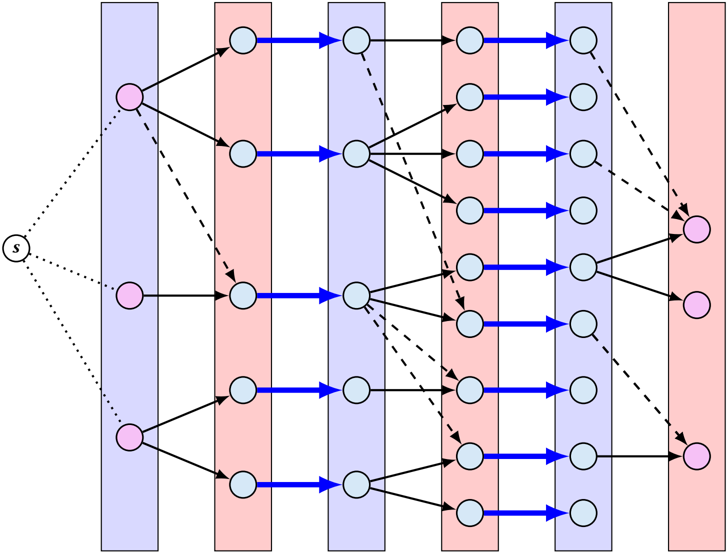
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