Mincost Flow

Problem Definition:

$$
\begin{array}{lll}
\text{min} & \sum_e c(e) f(e) & \\
\text{s.t.} & \forall e \in E: & 0 \le f(e) \le u(e) \\
 & \forall v \in V: & f(v) = b(v)
\end{array}
$$

Mincost Flow

Problem Definition:

$$\begin{aligned} \min \quad & \textstyle\sum_e c(e) f(e) \\ \text{s.t.} \quad & \forall e \in E: \quad 0 \le f(e) \le u(e) \\ & \forall v \in V: \quad f(v) = b(v) \end{aligned}$$

- $G = (V, E)$ is a directed graph.
- $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$ is the capacity function.
- $c : E \to \mathbb{R}$ is the cost function
  (note that $c(e)$ may be negative).
- $b : V \to \mathbb{R}$, $\sum_{v \in V} b(v) = 0$ is a demand function.

Mincost Flow

Problem Definition:

$$
\begin{aligned}
\min \quad & \textstyle\sum_e c(e)f(e) \\
\text{s.t.} \quad & \forall e \in E: \quad 0 \le f(e) \le u(e) \\
& \forall v \in V: \quad f(v) = b(v)
\end{aligned}
$$

- $G = (V, E)$ is a directed graph.
- $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$ is the capacity function.
- $c : E \to \mathbb{R}$ is the cost function
  (note that $c(e)$ may be negative).
- $b : V \to \mathbb{R}$, $\sum_{v \in V} b(v) = 0$ is a demand function.

Mincost Flow

Problem Definition:

$$\begin{aligned} \text{min} \quad & \textstyle\sum_e c(e) f(e) \\ \text{s.t.} \quad & \forall e \in E: \quad 0 \le f(e) \le u(e) \\ & \forall v \in V: \quad f(v) = b(v) \end{aligned}$$

- $G = (V, E)$ is a directed graph.
- $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$ is the capacity function.
- $c : E \to \mathbb{R}$ is the cost function (note that $c(e)$ may be negative).
- $b : V \to \mathbb{R}$, $\sum_{v \in V} b(v) = 0$ is a demand function.

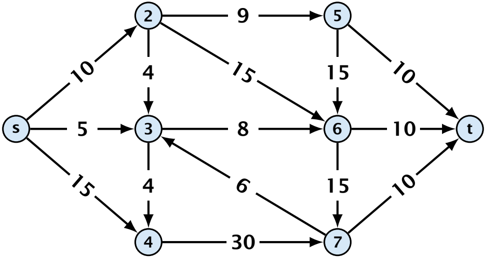
Mincost Flow

Problem Definition:

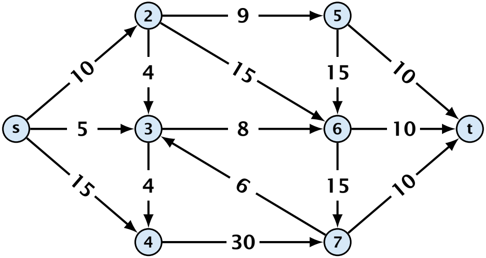
$$\begin{aligned} \min \quad & \textstyle\sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \quad 0 \le f(e) \le u(e) \\ & \forall v \in V: \quad f(v) = b(v) \end{aligned}$$

- $G = (V, E)$ is a directed graph.
- $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$ is the capacity function.
- $c : E \to \mathbb{R}$ is the cost function (note that $c(e)$ may be negative).
- $b : V \to \mathbb{R}$, $\sum_{v \in V} b(v) = 0$ is a demand function.

Solve Maxflow Using Mincost Flow

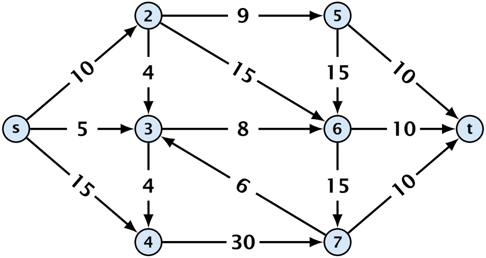


Solve Maxflow Using Mincost Flow



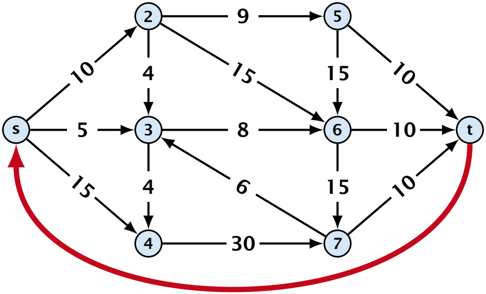
- Given a flow network for a standard maxflow problem.

Solve Maxflow Using Mincost Flow



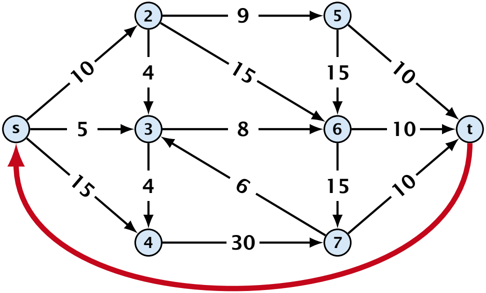
- Given a flow network for a standard maxflow problem.
- Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.

Solve Maxflow Using Mincost Flow



- Given a flow network for a standard maxflow problem.
- Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.
- Add an edge from t to s with infinite capacity and cost -1.

Solve Maxflow Using Mincost Flow



- Given a flow network for a standard maxflow problem.
- Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.
- Add an edge from t to s with infinite capacity and cost -1.
- Then, $\mathrm{val}(f^*) = -\mathrm{cost}(f_{\min})$, where f^* is a maxflow, and $f_{\min}$ is a mincost-flow.

Solve Maxflow Using Mincost Flow

Solve decision version of maxflow:

- Given a flow network for a standard maxflow problem, and a value k.
- Set $b(v) = 0$ for every node apart from s or t. Set $b(s) = -k$ and $b(t) = k$.
- Set edge-costs to zero, and keep the capacities.
- There exists a maxflow of value k if and only if the mincost-flow problem is feasible.

Solve Maxflow Using Mincost Flow

Solve decision version of maxflow:

- Given a flow network for a standard maxflow problem, and a value k.
- Set $b(v) = 0$ for every node apart from s or t. Set $b(s) = -k$ and $b(t) = k$.
- Set edge-costs to zero, and keep the capacities.
- There exists a maxflow of value k if and only if the mincost-flow problem is feasible.

Solve Maxflow Using Mincost Flow

Solve decision version of maxflow:

- Given a flow network for a standard maxflow problem, and a value k.
- Set $b(v) = 0$ for every node apart from s or t. Set $b(s) = -k$ and $b(t) = k$.
- Set edge-costs to zero, and keep the capacities.
- There exists a maxflow of value k if and only if the mincost-flow problem is feasible.

Solve Maxflow Using Mincost Flow

Solve decision version of maxflow:

- Given a flow network for a standard maxflow problem, and a value k.
- Set $b(v) = 0$ for every node apart from s or t. Set $b(s) = -k$ and $b(t) = k$.
- Set edge-costs to zero, and keep the capacities.
- There exists a maxflow of value k if and only if the mincost-flow problem is feasible.

Generalization

Our model:

$$\begin{array}{ll} \min & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e \in E: \quad 0 \le f(e) \le u(e) \\ & \forall v \in V: \quad f(v) = b(v) \end{array}$$

where $b : V \to \mathbb{R}$, $\sum_v b(v) = 0$; $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$; $c : E \to \mathbb{R}$;

A more general model?

$$\begin{array}{ll} \min & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e \in E: \quad \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: \quad a(v) \le f(v) \le b(v) \end{array}$$

where $a : V \to \mathbb{R}$, $b : V \to \mathbb{R}$; $\ell : E \to \mathbb{R} \cup \{-\infty\}$, $u : E \to \mathbb{R} \cup \{\infty\}$ $c : E \to \mathbb{R}$;

Generalization

Our model:

$$\begin{array}{lll} \text{min} & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & 0 \le f(e) \le u(e) \\ & \forall v \in V: & f(v) = b(v) \end{array}$$

where $b : V \to \mathbb{R}$, $\sum_v b(v) = 0$; $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$; $c : E \to \mathbb{R}$;

A more general model?

$$\begin{array}{lll} \text{min} & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: & a(v) \le f(v) \le b(v) \end{array}$$

where $a : V \to \mathbb{R}$, $b : V \to \mathbb{R}$; $\ell : E \to \mathbb{R} \cup \{-\infty\}$, $u : E \to \mathbb{R} \cup \{\infty\}$
$c : E \to \mathbb{R}$;

Generalization

Differences

- Flow along an edge e may have non-zero lower bound $\ell(e)$.
- Flow along e may have negative upper bound $u(e)$.
- The demand at a node v may have lower bound $a(v)$ and upper bound $b(v)$ instead of just lower bound = upper bound = $b(v)$.

Reduction I

$$
\begin{aligned}
\text{min} \quad & \textstyle\sum_e c(e) f(e) \\
\text{s.t.} \quad & \forall e \in E: \;\; \ell(e) \le f(e) \le u(e) \\
& \forall v \in V: \;\; a(v) \le f(v) \le b(v)
\end{aligned}
$$

We can assume that $a(v) = b(v)$:

Add new node r.

Add edge (r, v) for all $v \in V$.

Set $\ell(e) = c(e) = 0$ for these edges.

Set $u(e) = b(v) - a(v)$ for edge (r, v).

Set $a(v) = b(v)$ for all $v \in V$.

Set $b(r) = -\sum_{v \in V} b(v)$.

$-\sum_v b(v)$ is negative; hence r is only sending flow.

Reduction I

$$\begin{aligned} \min \quad & \textstyle\sum_e c(e) f(e) \\ \text{s.t.} \quad & \forall e \in E: \;\; \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: \;\; a(v) \le f(v) \le b(v) \end{aligned}$$

We can assume that $a(v) = b(v)$:

Add new node r.

Add edge (r, v) for all $v \in V$.

Set $\ell(e) = c(e) = 0$ for these edges.

Set $u(e) = b(v) - a(v)$ for edge (r, v).

Set $a(v) = b(v)$ for all $v \in V$.

Set $b(r) = -\sum_{v \in V} b(v)$.

$-\sum_v b(v)$ is negative; hence r is only sending flow.

Reduction I

$$\begin{aligned} \min \quad & \textstyle\sum_e c(e) f(e) \\ \text{s.t.} \quad & \forall e \in E: \ \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: \ a(v) \le f(v) \le b(v) \end{aligned}$$

We can assume that $a(v) = b(v)$:

Add new node r.

Add edge (r, v) for all $v \in V$.

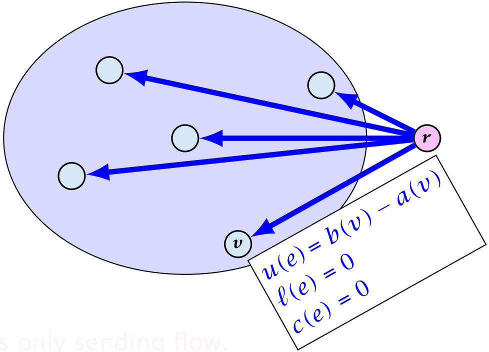
Set $\ell(e) = c(e) = 0$ for these edges.

Set $u(e) = b(v) - a(v)$ for edge (r, v).

Set $a(v) = b(v)$ for all $v \in V$.

Set $b(r) = -\sum_{v \in V} b(v)$.

$-\sum_v b(v)$ is negative; hence r is only sending flow.



Reduction I

$$\begin{array}{ll} \text{min} & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e\in E:\;\; \ell(e)\le f(e)\le u(e) \\ & \forall v\in V:\;\; a(v)\le f(v)\le b(v) \end{array}$$

We can assume that $a(v) = b(v)$:

Add new node r.

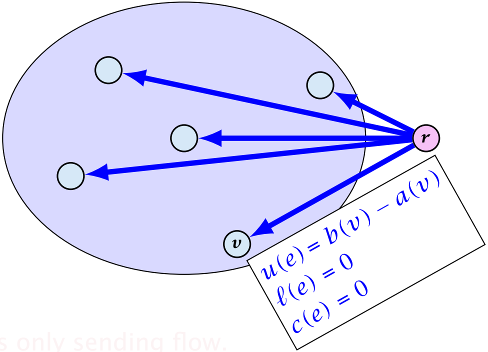
Add edge (r,v) for all $v\in V$.

Set $\ell(e)=c(e)=0$ for these edges.

Set $u(e)=b(v)-a(v)$ for edge (r,v).

Set $a(v)=b(v)$ for all $v\in V$.

Set $b(r)=-\sum_{v\in V} b(v)$.



$-\sum_v b(v)$ is negative; hence r is only sending flow.

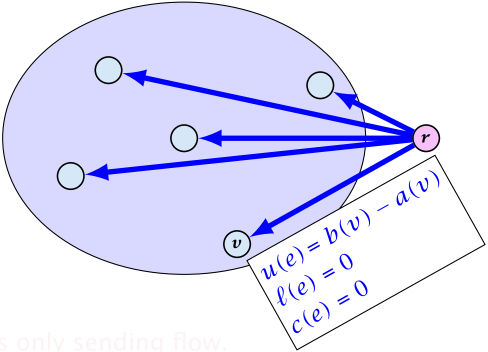
Reduction I

$$
\begin{aligned}
\min \quad & \textstyle\sum_e c(e) f(e) \\
\text{s.t.} \quad & \forall e \in E: \;\; \ell(e) \le f(e) \le u(e) \\
& \forall v \in V: \;\; a(v) \le f(v) \le b(v)
\end{aligned}
$$

We can assume that $a(v) = b(v)$:

Add new node r.

Add edge (r, v) for all $v \in V$.



Set $\ell(e) = c(e) = 0$ for these edges.

Set $u(e) = b(v) - a(v)$ for edge (r, v).

Set $a(v) = b(v)$ for all $v \in V$.

Set $b(r) = -\sum_{v \in V} b(v)$.

$-\sum_v b(v)$ is negative; hence r is only sending flow.

Reduction I

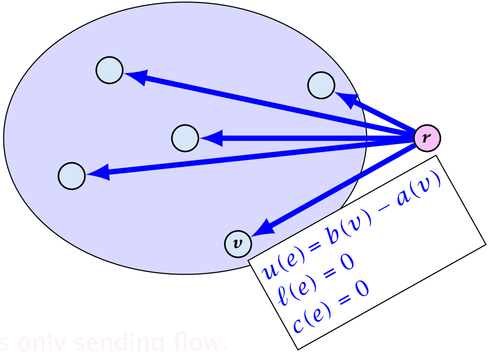
$$\begin{array}{ll} \min & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e \in E: \;\; \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: \;\; a(v) \le f(v) \le b(v) \end{array}$$

We can assume that $a(v) = b(v)$:

Add new node r.

Add edge (r, v) for all $v \in V$.

Set $\ell(e) = c(e) = 0$ for these edges.



Set $u(e) = b(v) - a(v)$ for edge (r, v).

Set $a(v) = b(v)$ for all $v \in V$.

Set $b(r) = -\sum_{v \in V} b(v)$.

$-\sum_v b(v)$ is negative; hence r is only sending flow.

Reduction I

$$\begin{aligned} \min \quad & \textstyle\sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \;\; \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: \;\; a(v) \le f(v) \le b(v) \end{aligned}$$

We can assume that $a(v) = b(v)$:

Add new node r.

Add edge (r, v) for all $v \in V$.

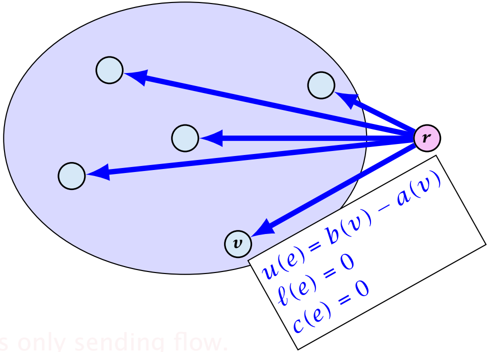
Set $\ell(e) = c(e) = 0$ for these edges.

Set $u(e) = b(v) - a(v)$ for edge (r, v).

Set $a(v) = b(v)$ for all $v \in V$.

Set $b(r) = -\sum_{v \in V} b(v)$.

$-\sum_v b(v)$ is negative; hence r is only sending flow.



Reduction I

$$\begin{aligned} \min \quad & \textstyle\sum_e c(e) f(e) \\ \text{s.t.} \quad & \forall e \in E: \quad \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: \quad a(v) \le f(v) \le b(v) \end{aligned}$$

We can assume that $a(v) = b(v)$:

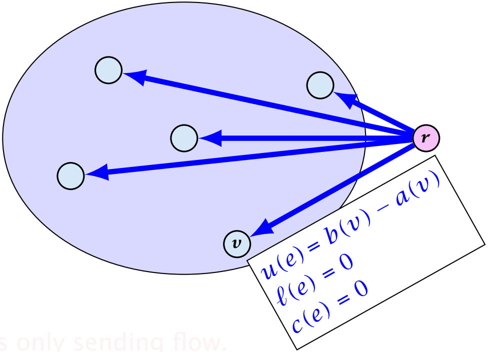
Add new node r.

Add edge (r, v) for all $v \in V$.

Set $\ell(e) = c(e) = 0$ for these edges.

Set $u(e) = b(v) - a(v)$ for edge (r, v).

Set $a(v) = b(v)$ for all $v \in V$.



Set $b(r) = -\sum_{v \in V} b(v)$.

$-\sum_v b(v)$ is negative; hence r is only sending flow.

Reduction I

$$\begin{aligned} \min \quad & \textstyle\sum_e c(e)f(e) \\ \text{s.t.} \quad & \forall e \in E: \; \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: \; a(v) \le f(v) \le b(v) \end{aligned}$$

We can assume that $a(v) = b(v)$:

Add new node r.

Add edge (r, v) for all $v \in V$.

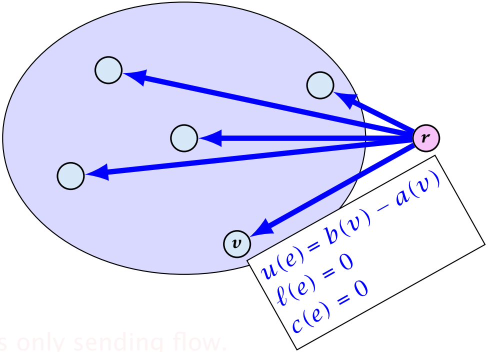
Set $\ell(e) = c(e) = 0$ for these edges.

Set $u(e) = b(v) - a(v)$ for edge (r, v).

Set $a(v) = b(v)$ for all $v \in V$.

Set $b(r) = -\sum_{v \in V} b(v)$.

$-\sum_v b(v)$ is negative; hence r is only sending flow.



Reduction I

$$\begin{array}{lll} \min & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: & a(v) \le f(v) \le b(v) \end{array}$$

We can assume that $a(v) = b(v)$:

Add new node r.

Add edge (r, v) for all $v \in V$.

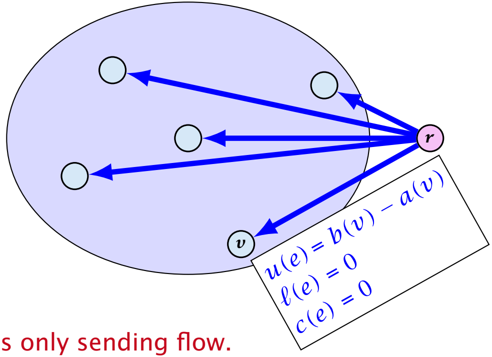
Set $\ell(e) = c(e) = 0$ for these edges.

Set $u(e) = b(v) - a(v)$ for edge (r, v).

Set $a(v) = b(v)$ for all $v \in V$.

Set $b(r) = -\sum_{v \in V} b(v)$.

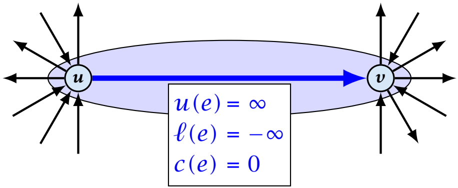
$-\sum_v b(v)$ is negative; hence r is only sending flow.



Reduction II

$$\begin{array}{ll} \min & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e \in E: \quad \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: \quad f(v) = b(v) \end{array}$$

We can assume that either $\ell(e) \neq -\infty$ or $u(e) \neq \infty$:



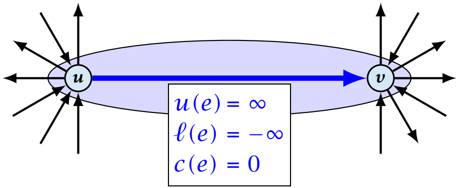
If $c(e) = 0$ we can contract the edge/identify nodes u and v.

If $c(e) \neq 0$ we can transform the graph so that $c(e) = 0$.

Reduction II

$$\begin{array}{lll} \min & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: & f(v) = b(v) \end{array}$$

We can assume that either $\ell(e) \neq -\infty$ or $u(e) \neq \infty$:



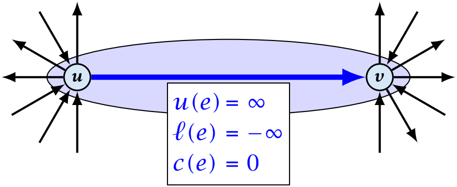
If $c(e) = 0$ we can contract the edge/identify nodes u and v.

If $c(e) \neq 0$ we can transform the graph so that $c(e) = 0$.

Reduction II

$$\begin{aligned} \min \quad & \textstyle\sum_e c(e) f(e) \\ \text{s.t.} \quad & \forall e \in E: \quad \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: \quad f(v) = b(v) \end{aligned}$$

We can assume that either $\ell(e) \neq -\infty$ or $u(e) \neq \infty$:

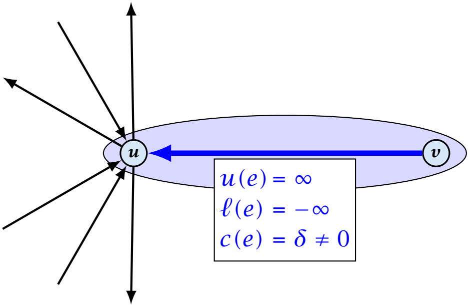


If $c(e) = 0$ we can contract the edge/identify nodes u and v.

If $c(e) \neq 0$ we can transform the graph so that $c(e) = 0$.

Reduction II

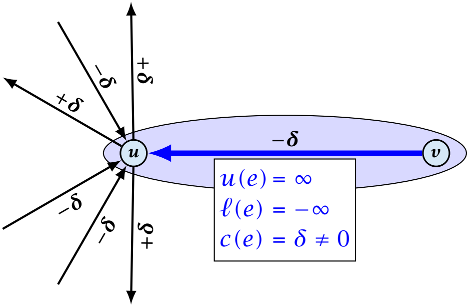
We can transform any network so that a particular edge has cost $c(e) = 0$:



Additionally we set $b(u) = 0$.

Reduction II

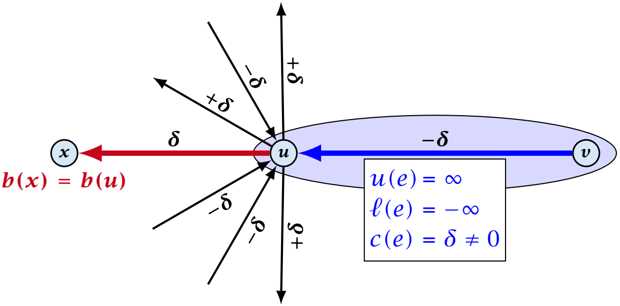
We can transform any network so that a particular edge has cost $c(e) = 0$:



Additionally we set $b(u) = 0$.

Reduction II

We can transform any network so that a particular edge has cost $c(e) = 0$:

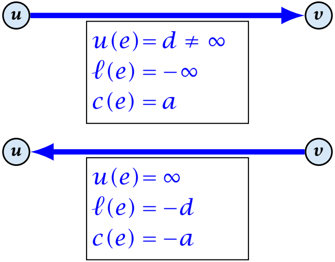


Additionally we set $b(u) = 0$.

Reduction III

$$\begin{array}{lll} \min & \sum_e c(e) f(e) & \\ \text{s.t.} & \forall e \in E: & \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: & f(v) = b(v) \end{array}$$

We can assume that $\ell(e) \neq -\infty$:

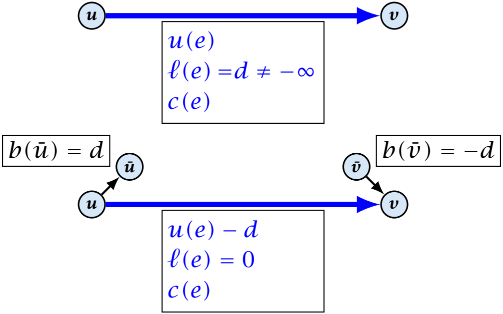


Replace the edge by an edge in opposite direction.

Reduction IV

$$\begin{array}{lll} \min & \sum_e c(e)f(e) & \\ \text{s.t.} & \forall e \in E: & \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: & f(v) = b(v) \end{array}$$

We can assume that $\ell(e) = 0$:



The added edges have infinite capacity and cost $c(e)/2$.

Applications

Caterer Problem

- She needs to supply r_i napkins on N successive days.
- She can buy new napkins at p cents each.
- She can launder them at a fast laundry that takes m days and cost f cents a napkin.
- She can use a slow laundry that takes $k > m$ days and costs s cents each.
- At the end of each day she should determine how many to send to each laundry and how many to buy in order to fulfill demand.
- Minimize cost.

Applications

Caterer Problem

- She needs to supply r_i napkins on N successive days.
- She can buy new napkins at p cents each.
- She can launder them at a fast laundry that takes m days and cost f cents a napkin.
- She can use a slow laundry that takes $k > m$ days and costs s cents each.
- At the end of each day she should determine how many to send to each laundry and how many to buy in order to fulfill demand.
- Minimize cost.

Applications

Caterer Problem

- She needs to supply r_i napkins on N successive days.
- She can buy new napkins at p cents each.
- She can launder them at a fast laundry that takes m days and cost f cents a napkin.
- She can use a slow laundry that takes $k > m$ days and costs s cents each.
- At the end of each day she should determine how many to send to each laundry and how many to buy in order to fulfill demand.
- Minimize cost.

Applications

Caterer Problem

- She needs to supply r_i napkins on N successive days.
- She can buy new napkins at p cents each.
- She can launder them at a fast laundry that takes m days and cost f cents a napkin.
- She can use a slow laundry that takes $k > m$ days and costs s cents each.
- At the end of each day she should determine how many to send to each laundry and how many to buy in order to fulfill demand.
- Minimize cost.

Applications

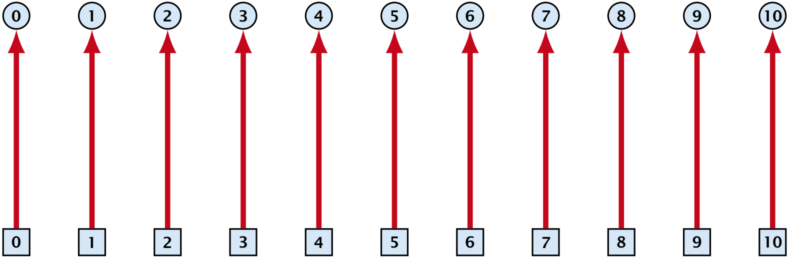
Caterer Problem

- She needs to supply r_i napkins on N successive days.
- She can buy new napkins at p cents each.
- She can launder them at a fast laundry that takes m days and cost f cents a napkin.
- She can use a slow laundry that takes $k > m$ days and costs s cents each.
- At the end of each day she should determine how many to send to each laundry and how many to buy in order to fulfill demand.
- Minimize cost.

Applications

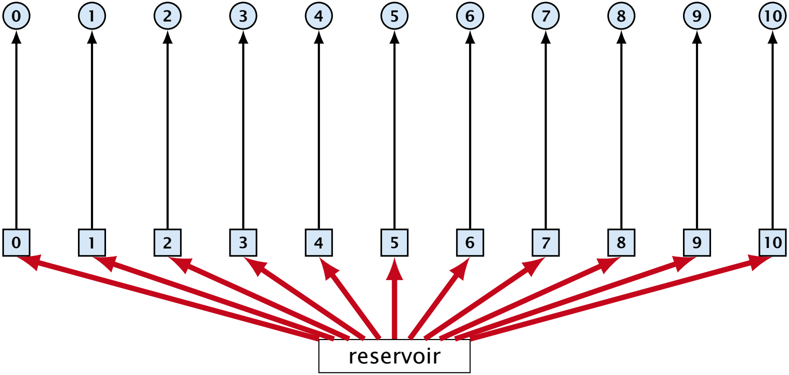
Caterer Problem

- She needs to supply r_i napkins on N successive days.
- She can buy new napkins at p cents each.
- She can launder them at a fast laundry that takes m days and cost f cents a napkin.
- She can use a slow laundry that takes $k > m$ days and costs s cents each.
- At the end of each day she should determine how many to send to each laundry and how many to buy in order to fulfill demand.
- Minimize cost.



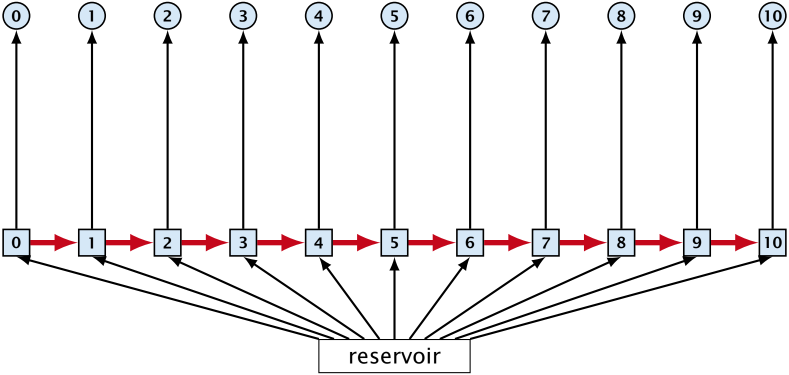
day edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = r_i$;
cost: $c(e) = 0$



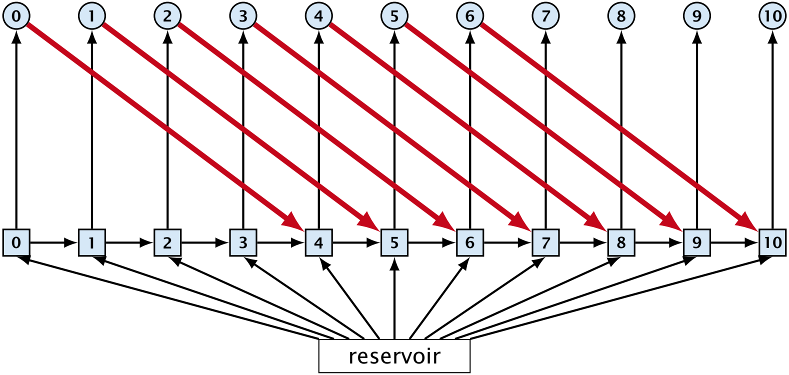
buy edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = p$



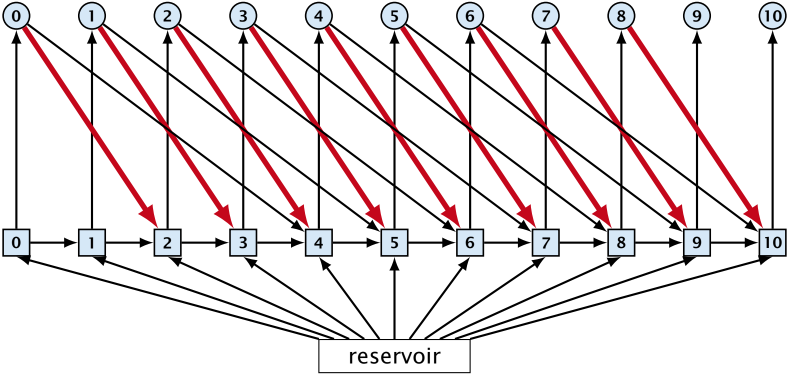
forward edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = 0$



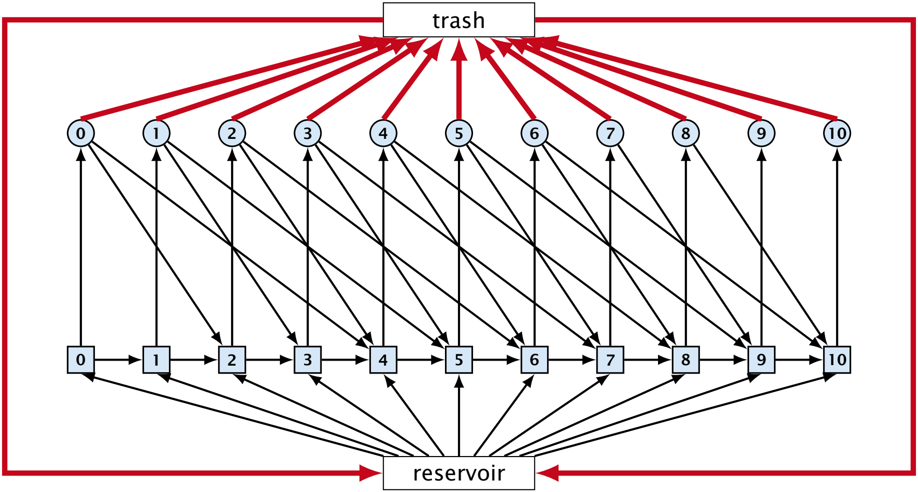
slow edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = s$



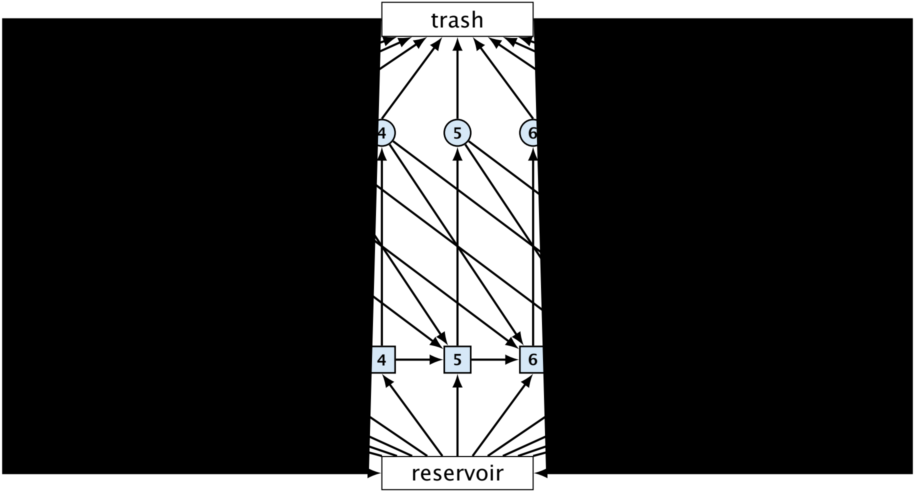
fast edges:

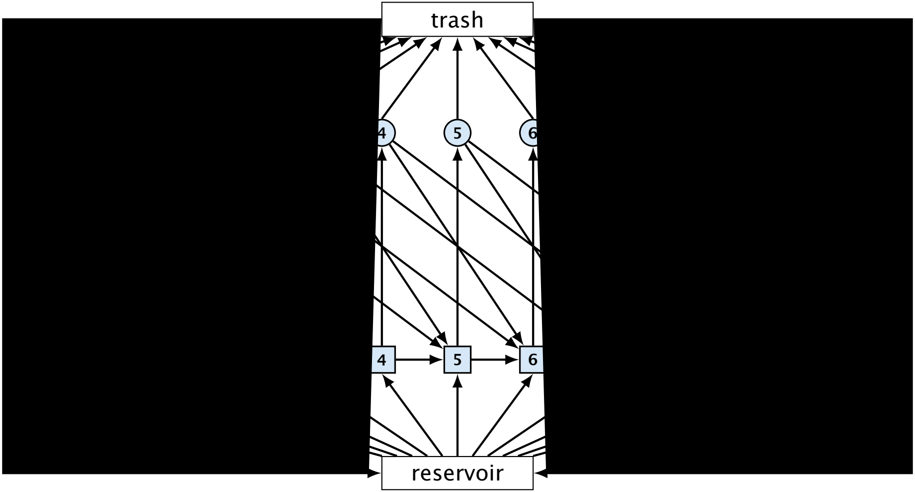
upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = f$



trash edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = 0$;
cost: $c(e) = 0$


trash
4
5
6
4
5
6
reservoir


trash
4
5
6
4
5
6
reservoir

Residual Graph

The residual graph for a mincost flow is exactly defined as the residual graph for standard flows, with the only exception that one needs to define a cost for the residual edge.

For a flow of z from u to v the residual edge (v, u) has capacity z and a cost of $-c((u, v))$.

Residual Graph

The residual graph for a mincost flow is exactly defined as the residual graph for standard flows, with the only exception that one needs to define a cost for the residual edge.

For a flow of z from u to v the residual edge (v, u) has capacity z and a cost of $-c((u, v))$.

15 Mincost Flow

A circulation in a graph $G = (V, E)$ is a function $f : E \to \mathbb{R}^+$ that has an excess flow $f(v) = 0$ for every node $v \in V$.

A circulation is feasible if it fulfills capacity constraints, i.e., $f(e) \leq u(e)$ for every edge of G.

15 Mincost Flow

A circulation in a graph $G = (V, E)$ is a function $f : E \to \mathbb{R}^+$ that has an excess flow $f(v) = 0$ for every node $v \in V$.

A circulation is feasible if it fulfills capacity constraints, i.e., $f(e) \leq u(e)$ for every edge of G.

Lemma 1

A given flow is a mincost-flow if and only if the corresponding residual graph G_f does not have a feasible circulation of negative cost.

Suppose that g is a feasible circulation of negative cost in the residual graph.

Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

Lemma 1

A given flow is a mincost-flow if and only if the corresponding residual graph G_f does not have a feasible circulation of negative cost.

⇒ Suppose that g is a feasible circulation of negative cost in the residual graph.

Then $f + g$ is a feasible flow with cost $\text{cost}(f) + \text{cost}(g) < \text{cost}(f)$. Hence, f is not minimum cost.

⇐ Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

Clearly $f^* - f$ is a circulation of negative cost. One can also easily see that it is feasible for the residual graph. (after sending $-f$ in the residual graph (pushing all flow back) we arrive at the original graph; for this f^* is clearly feasible)

Lemma 1

A given flow is a mincost-flow if and only if the corresponding residual graph G_f does not have a feasible circulation of negative cost.

⇒ Suppose that g is a feasible circulation of negative cost in the residual graph.

Then $f + g$ is a feasible flow with cost $\mathrm{cost}(f) + \mathrm{cost}(g) < \mathrm{cost}(f)$. Hence, f is not minimum cost.

⇐ Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

Clearly $f^* - f$ is a circulation of negative cost. One can also easily see that it is feasible for the residual graph. (after sending $-f$ in the residual graph (pushing all flow back) we arrive at the original graph; for this f^* is clearly feasible)

Lemma 1

A given flow is a mincost-flow if and only if the corresponding residual graph G_f does not have a feasible circulation of negative cost.

⇒ Suppose that g is a feasible circulation of negative cost in the residual graph.

Then $f + g$ is a feasible flow with cost $\text{cost}(f) + \text{cost}(g) < \text{cost}(f)$. Hence, f is not minimum cost.

⇐ Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

Clearly $f^* - f$ is a circulation of negative cost. One can also easily see that it is feasible for the residual graph. (after sending $-f$ in the residual graph (pushing all flow back) we arrive at the original graph; for this f^* is clearly feasible)

Lemma 1

A given flow is a mincost-flow if and only if the corresponding residual graph G_f does not have a feasible circulation of negative cost.

- ⇒ Suppose that g is a feasible circulation of negative cost in the residual graph.

  Then $f + g$ is a feasible flow with cost $\text{cost}(f) + \text{cost}(g) < \text{cost}(f)$. Hence, f is not minimum cost.

- ⇐ Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

  Clearly $f^* - f$ is a circulation of negative cost. One can also easily see that it is feasible for the residual graph. (after sending $-f$ in the residual graph (pushing all flow back) we arrive at the original graph; for this f^* is clearly feasible)

15 Mincost Flow

Lemma 2

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

Proof.

- Suppose that we have a negative cost circulation.
- Find directed path only using edges that have non-zero flow.
- If this path has negative cost you are done.
- Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.
- You still have a circulation with negative cost.
- Repeat.

Lemma 2

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

Proof.

- Suppose that we have a negative cost circulation.
- Find directed path only using edges that have non-zero flow.
- If this path has negative cost you are done.
- Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.
- You still have a circulation with negative cost.
- Repeat.

15 Mincost Flow

Lemma 2

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

Proof.

- Suppose that we have a negative cost circulation.
- Find directed path only using edges that have non-zero flow.
- If this path has negative cost you are done.
- Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.
- You still have a circulation with negative cost.
- Repeat.

15 Mincost Flow

Lemma 2

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

Proof.

- Suppose that we have a negative cost circulation.
- Find directed path only using edges that have non-zero flow.
- If this path has negative cost you are done.
- Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.
- You still have a circulation with negative cost.
- Repeat.

15 Mincost Flow

Lemma 2

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

Proof.

- Suppose that we have a negative cost circulation.
- Find directed path only using edges that have non-zero flow.
- If this path has negative cost you are done.
- Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.
- You still have a circulation with negative cost.
- Repeat.

15 Mincost Flow

Lemma 2

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

Proof.

- Suppose that we have a negative cost circulation.
- Find directed path only using edges that have non-zero flow.
- If this path has negative cost you are done.
- Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.
- You still have a circulation with negative cost.
- Repeat.

15 Mincost Flow

Lemma 2

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

Proof.

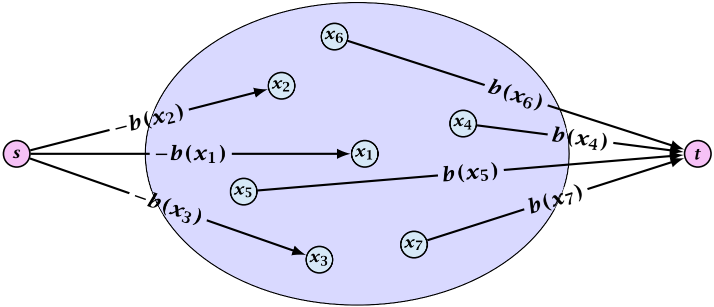
- Suppose that we have a negative cost circulation.
- Find directed path only using edges that have non-zero flow.
- If this path has negative cost you are done.
- Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.
- You still have a circulation with negative cost.
- Repeat.

15 Mincost Flow

Algorithm 49 CycleCanceling$(G = (V, E), c, u, b)$

1: establish a feasible flow f in G
2: **while** G_f contains negative cycle **do**
3: use Bellman-Ford to find a negative circuit Z
4: $\delta \leftarrow \min\{u_f(e) \mid e \in Z\}$
5: augment δ units along Z and update G_f

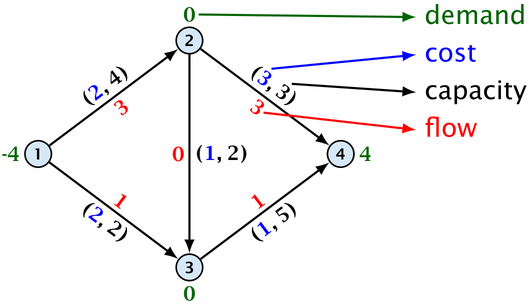
How do we find the initial feasible flow?



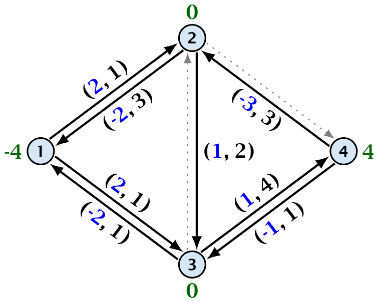
- Connect new node s to all nodes with negative $b(v)$-value.
- Connect nodes with positive $b(v)$-value to a new node t.
- There exist a feasible flow in the original graph iff in the resulting graph there exists an s-t flow of value

$$\sum_{v:b(v)<0} (-b(v)) = \sum_{v:b(v)>0} b(v) \ .$$

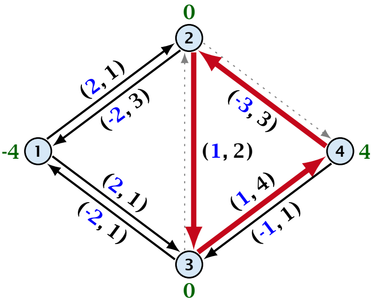
15 Mincost Flow



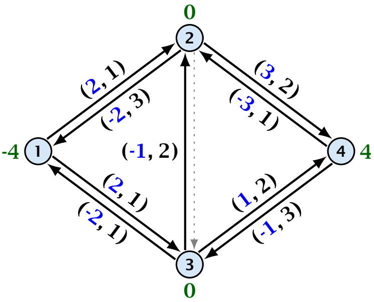
15 Mincost Flow



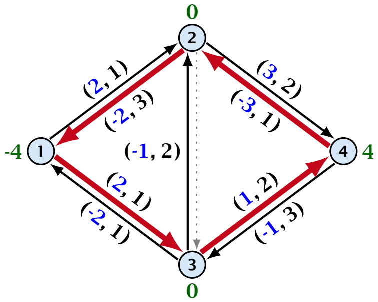
15 Mincost Flow



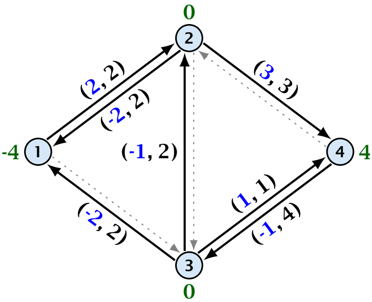
15 Mincost Flow



15 Mincost Flow



15 Mincost Flow



15 Mincost Flow

Lemma 3
The improving cycle algorithm runs in time $\mathcal{O}(nm^2CU)$, for integer capacities and costs, when for all edges e, $|c(e)| \leq C$ and $|u(e)| \leq U$.

- Running time of Bellman-Ford is $\mathcal{O}(mn)$.
- Pushing flow along the cycle can be done in time $\mathcal{O}(m)$.
- Each iteration decreases the total cost by at least 1.
- The true optimum cost must lie in the interval $[-CU,\dots,+CU]$.

Note that this lemma is weak since it does not allow for edges with infinite capacity.

15 Mincost Flow

A general mincost flow problem is of the following form:

$$\begin{array}{ll} \text{min} & \sum_e c(e)f(e) \\ \text{s.t.} & \forall e \in E: \;\; \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: \;\; a(v) \le f(v) \le b(v) \end{array}$$

where $a : V \to \mathbb{R}$, $b : V \to \mathbb{R}$; $\ell : E \to \mathbb{R} \cup \{-\infty\}$, $u : E \to \mathbb{R} \cup \{\infty\}$ $c : E \to \mathbb{R}$;

Lemma 4 (without proof)

A general mincost flow problem can be solved in polynomial time.