6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

This is the general form of a linear recurrence relation of order k with constant coefficients ($c_0, c_k \neq 0$).

- $T(n)$ only depends on the k preceding values. This means the recurrence relation is of order k.
- The recurrence is linear as there are no products of $T[n]$'s.
- If $f(n) = 0$ then the recurrence relation becomes a linear, homogenous recurrence relation of order k.

6.3 The Characteristic Polynomial

Observations:

- The solution $T[0], T[1], T[2], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[0], \ldots, T[k-1]$.
- In fact, any k consecutive values completely determine the solution.
- k non-concecutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

- First determine all solutions that satisfy recurrence relation.
- Then pick the right one by analyzing boundary conditions.
- First consider the homogenous case.

The Homogenous Case

The solution space

$$S = \left\{T = T[0], T[1], T[2], \ldots \mid T \text{ fulfills recurrence relation}\right\}$$

is a vector space. This means that if $T_1, T_2 \in S$, then also $\alpha T_1 + \beta T_2 \in S$, for arbitrary constants α, β.

How do we find a non-trivial solution?

We guess that the solution is of the form λ^n, $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \cdots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

The Homogenous Case

Dividing by λ^{n-k} gives that all these constraints are identical to

$$\underbrace{c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \cdots + c_k}_{\text{characteristic polynomial } P[\lambda]} = 0$$

This means that if λ_i is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \ldots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_k\lambda_k^n$$

is a solution for arbitrary values α_i.

The Homogenous Case

Lemma 5

Assume that the characteristic polynomial has k distinct roots $\lambda_1, \dots, \lambda_k$. Then all solutions to the recurrence relation are of the form

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_k\lambda_k^n \ .$$

Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \dots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i' s$ such that these conditions are met:

$$\begin{array}{ccccccccc} \alpha_1\cdot\lambda_1 & + & \alpha_2\cdot\lambda_2 & + & \cdots & + & \alpha_k\cdot\lambda_k & = & T[1] \\ \alpha_1\cdot\lambda_1^2 & + & \alpha_2\cdot\lambda_2^2 & + & \cdots & + & \alpha_k\cdot\lambda_k^2 & = & T[2] \\ & & & & \vdots & & & & \\ \alpha_1\cdot\lambda_1^k & + & \alpha_2\cdot\lambda_2^k & + & \cdots & + & \alpha_k\cdot\lambda_k^k & = & T[k] \end{array}$$

The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i' s$ such that these conditions are met:

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ & \vdots & & \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_k^k \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} T[1] \\ T[2] \\ \vdots \\ T[k] \end{pmatrix}$$

We show that the column vectors are linearly independent. Then the above equation has a solution.

The Homogenous Case

Proof (cont.).

This we show by induction:

- **base case** ($k = 1$):
  A vector (λ_i), $\lambda_i \neq 0$ is linearly independent.
- **induction step** ($k \to k+1$):
  assume for contradiction that there exist α_i's with

$$\alpha_1 \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_1^{k-1} \\ \lambda_1^k \end{pmatrix} + \cdots + \alpha_k \begin{pmatrix} \lambda_k \\ \vdots \\ \lambda_k^{k-1} \\ \lambda_k^k \end{pmatrix} = 0$$

and not all $\alpha_i = 0$. **Then all $\boldsymbol{\alpha_i \neq 0}$!** □

The Homogeneous Case

$$\alpha_1 \underbrace{\begin{pmatrix} \lambda_1 \\ \lambda_1^2 \\ \vdots \\ \lambda_1^{k-1} \\ \lambda_1^k \end{pmatrix}}_{v_1 := \ldots,\ \lambda_1 v_1 = \ldots} + \cdots + \alpha_k \underbrace{\begin{pmatrix} \lambda_1 \\ \lambda_k^2 \\ \vdots \\ \lambda_k^{k-1} \\ \lambda_k^k \end{pmatrix}}_{v_k := \ldots,\ \lambda_k v_k = \ldots} = 0$$

This means that

$$\sum_{i=1}^{k} \alpha_i v_i = 0 \text{ and } \sum_{i=1}^{k} \lambda_i \alpha_i v_i = 0$$

Hence,

$$\sum_{i=1}^{k-1} \alpha_i v_i + \alpha_k v_k = 0 \text{ and } -\frac{1}{\lambda_k} \sum_{i=1}^{k-1} \lambda_i \alpha_i v_i = \alpha_k v_k$$

The Homogeneous Case

This gives that

$$\sum_{i=1}^{k-1} (1 - \frac{\lambda_i}{\lambda_k}) \alpha_i v_i = 0 \ .$$

This is a contradiction as the v_i's are linearly independent because of induction hypothesis.

The Homogeneous Case

What happens if the roots are not all distinct?

Suppose we have a root λ_i with multiplicity (Vielfachheit) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^n$.

To see this consider the polynomial

$$P(\lambda)\lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

Since λ_i is a root we can write this as $Q(\lambda)(\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root λ_i.

This means

$$c_0 n\lambda_i^{n-1} + c_1(n-1)\lambda_i^{n-2} + \cdots + c_k(n-k)\lambda_i^{n-k-1} = 0$$

Hence,

$$\underbrace{c_0 n\lambda_i^n}_{T[n]} + \underbrace{c_1(n-1)\lambda_i^{n-1}}_{T[n-1]} + \cdots + \underbrace{c_k(n-k)\lambda_i^{n-k}}_{T[n-k]} = 0$$

The Homogeneous Case

Suppose λ_i has multiplicity j. We know that

$$c_0 n\lambda_i^n + c_1(n-1)\lambda_i^{n-1} + \cdots + c_k(n-k)\lambda_i^{n-k} = 0$$

(after taking the derivative; multiplying with λ; plugging in λ_i)

Doing this again gives

$$c_0 n^2\lambda_i^n + c_1(n-1)^2\lambda_i^{n-1} + \cdots + c_k(n-k)^2\lambda_i^{n-k} = 0$$

We can continue $j-1$ times.

Hence, $n^\ell\lambda_i^n$ is a solution for $\ell \in 0, \ldots, j-1$.

The Homogeneous Case

Lemma 6
Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n-1] + \cdots + c_kT[n-k] = 0$$

Let λ_i, $i = 1, \dots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i. Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i - 1} \alpha_{ij} \cdot (n^j \lambda_i^n) \ .$$

The full proof is omitted. We have only shown that any choice of α_{ij}'s is a solution to the recurrence.

Example: Fibonacci Sequence

$$T[0] = 0$$
$$T[1] = 1$$
$$T[n] = T[n-1] + T[n-2] \text{ for } n \geq 2$$

The characteristic polynomial is

$$\lambda^2 - \lambda - 1$$

Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$$

Example: Fibonacci Sequence

Hence, the solution is of the form

$$\alpha \left(\frac{1+\sqrt{5}}{2}\right)^n + \beta \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$T[0] = 0$ gives $\alpha + \beta = 0$.

$T[1] = 1$ gives

$$\alpha \left(\frac{1+\sqrt{5}}{2}\right) + \beta \left(\frac{1-\sqrt{5}}{2}\right) = 1 \Longrightarrow \alpha - \beta = \frac{2}{\sqrt{5}}$$

Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right]$$

The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving homogeneous, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) \ ,$$

where T_h is any solution to the homogeneous equation, and T_p is one particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.

The Inhomogeneous Case

Example:

$$T[n] = T[n-1] + 1 \qquad T[0] = 1$$

Then,

$$T[n-1] = T[n-2] + 1 \qquad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n-1] = T[n-1] - T[n-2] \qquad (n \geq 2)$$

or

$$T[n] = 2T[n-1] - T[n-2] \qquad (n \geq 2)$$

I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$.

The Inhomogeneous Case

Example: Characteristic polynomial:

$$\underbrace{\lambda^2 - 2\lambda + 1}_{(\lambda-1)^2} = 0$$

Then the solution is of the form

$$T[n] = \alpha 1^n + \beta n 1^n = \alpha + \beta n$$

$T[0] = 1$ gives $\alpha = 1$.

$T[1] = 2$ gives $1 + \beta = 2 \Longrightarrow \beta = 1$.

The Inhomogeneous Case

If $f(n)$ is a polynomial of degree r this method can be applied $r+1$ times to obtain a homogeneous equation:

$$T[n] = T[n-1] + n^2$$

Shift:

$$T[n-1] = T[n-2] + (n-1)^2 = T[n-2] + n^2 - 2n + 1$$

Difference:

$$T[n] - T[n-1] = T[n-1] - T[n-2] + 2n - 1$$

$$T[n] = 2T[n-1] - T[n-2] + 2n - 1$$

$$T[n] = 2T[n-1] - T[n-2] + 2n - 1$$

Shift:

$$\begin{aligned} T[n-1] &= 2T[n-2] - T[n-3] + 2(n-1) - 1 \\ &= 2T[n-2] - T[n-3] + 2n - 3 \end{aligned}$$

Difference:

$$\begin{aligned} T[n] - T[n-1] =& 2T[n-1] - T[n-2] + 2n - 1 \\ & - 2T[n-2] + T[n-3] - 2n + 3 \end{aligned}$$

$$T[n] = 3T[n-1] - 3T[n-2] + T[n-3] + 2$$

and so on...

6.4 Generating Functions

Definition 7 (Generating Function)

Let $(a_n)_{n\ge 0}$ be a sequence. The corresponding

- generating function (Erzeugendenfunktion) is

$$F(z) := \sum_{n=0}^{\infty} a_n z^n;$$

- exponential generating function (exponentielle Erzeugendenfunktion) is

$$F(z) = \sum_{n\ge 0} \frac{a_n}{n!} z^n.$$

6.4 Generating Functions

Example 8

1. The generating function of the sequence $(1,0,0,\dots)$ is

$$F(z) = 1\,.$$

2. The generating function of the sequence $(1,1,1,\dots)$ is

$$F(z) = \frac{1}{1-z}\,.$$