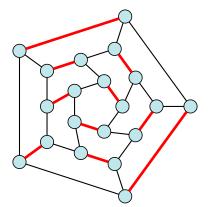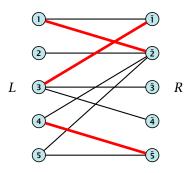# Matching

- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Maximum Matching: find a matching of maximum cardinality

Bipartite Matching

- Input: undirected, bipartite graph $G = (L \uplus R, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
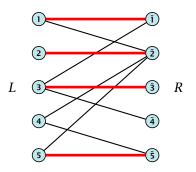- Maximum Matching: find a matching of maximum cardinality

Bipartite Matching

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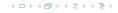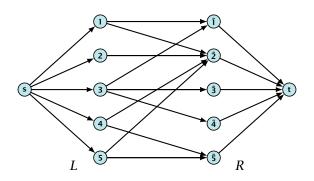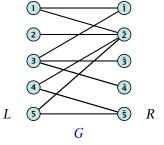- Maximum Matching: find a matching of maximum cardinality

Maxflow Formulation

- ▶ Input: undirected, bipartite graph $G = (L \uplus R \uplus \{s, t\}, E')$.
- ▶ Direct all edges from L to R.
- ▶ Add source s and connect it to all nodes on the left.
- ▶ Add t and connect all nodes on the right to t.
- ▶ All edges have unit capacity.

Proof

Max cardinality matching in $G \leq$ value of maxflow in G'

- ▶ Given a maximum matching M of cardinality k.
- ▶ Consider flow f that sends one unit along each of k paths.
- ▶ f is a flow and has cardinality k.

Proof

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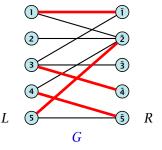**Max cardinality matching in $G \leq$ value of maxflow in G'**

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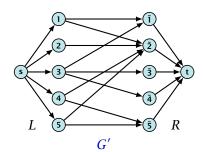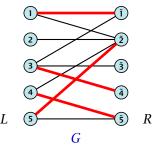**Max cardinality matching in G ≤ value of maxflow in G'**

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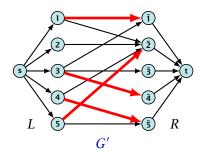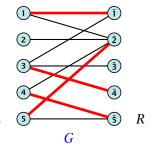- ▶ f is a flow and has cardinality k.

Proof

Max cardinality matching in $G \geq$ value of maxflow in G'

- Let f be a maxflow in G' of value k
- Integrality theorem $\Rightarrow k$ integral; we can assume f is 0/1.
- Consider $M=$ set of edges from L to R with $f(e) = 1$.
- Each node in L and R participates in at most one edge in M.
- $|M| = k$, as the flow must use at least k middle edges.

G' $\qquad$ G

Proof

Max cardinality matching in G ≥ value of maxflow in G'

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G'

G

Proof

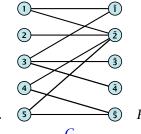**Max cardinality matching in $G \geq$ value of maxflow in G'**

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G' G

Proof

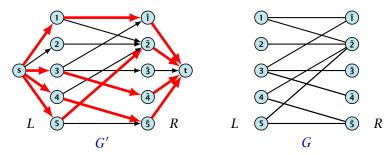**Max cardinality matching in G ≥ value of maxflow in G'**
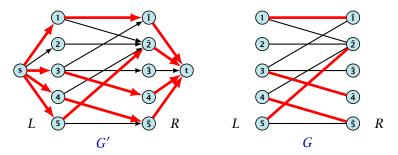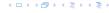
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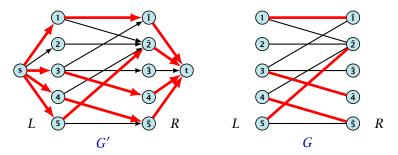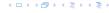

G' G

14.1 Matching

Which flow algorithm to use?

- ▸ Generic augmenting path: $\mathcal{O}(m \operatorname{val}(f^*)) = \mathcal{O}(mn)$.
- ▸ Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.

Baseball Elimination

team	wins	losses	remaining games			
i	w_i	ℓ_i	Atl	Phi	NY	Mon
Atlanta	83	71	-	1	6	1
Philadelphia	80	79	1	-	0	2
New York	78	78	6	0	-	0
Montreal	77	82	1	2	0	-

Which team can end the season with most wins?

▶ Montreal is eliminated, since even after winning all remaining games there are only 80 wins.

▶ But also Philadelphia is eliminated. Why?

Baseball Elimination

Formal definition of the problem:

- ▶ Given a set S of teams, and one specific team $z \in S$.
- ▶ Team x has already won w_x games.
- ▶ Team x still has to play team y, r_{xy} times.
- ▶ Does team z still have a chance to finish with the most number of wins.

Baseball Elimination

Flow networks for $z = 3$. M is number of wins Team 3 can still obtain.

Idea. Distribute the results of remaining games in such a way that no team gets too many wins.

Certificate of Elimination

Let $T \subseteq S$ be a subset of teams. Define

$$w(T) := \sum_{i \in T} w_i, \qquad r(T) := \sum_{i,j \in T, i < j} r_{ij}$$

wins of
teams in T

remaining games
among teams in T

If $\frac{w(T) + r(T)}{|T|} > M$ then one of the teams in T will have more than M wins in the end. A team that can win at most M games is therefore eliminated.

Theorem 83

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij \in S \setminus \{z\}, i<j} r_{ij}$.

Proof ($\Leftarrow$)

- Consider the mincut A in the flow network. Let T be the set of team-nodes in A.

- If for a node x-y not both team nodes x and y are in T, then x-$y \notin A$ as otw. the cut would cut an infinite capacity edge.

- We don't find a flow that saturates all source edges:

$$r(S \setminus \{z\}) > \mathrm{cap}(S, V \setminus S)$$

$$\geq \sum_{i<j:\, i \notin T \vee j \notin T} r_{ij} + \sum_{i \in T} (M - w_i)$$

$$\geq r(S \setminus \{z\}) - r(T) + |T|M - w(T)$$

- This gives $M < (w(T) + r(T))/|T|$, i.e., z is eliminated.

Baseball Elimination

Proof (⇒)

- Suppose we have a flow that saturates all source edges.
- We can assume that this flow is integral.
- For every pairing x-y it defines how many games team x and team y should win.
- The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.
- This is less than $M - w_x$ because of capacity constraints.
- Hence, we found a set of results for the remaining games, such that no team obtains more than M wins in total.
- Hence, team z is not eliminated.

Project Selection

Project selection problem:

- Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).

- Some projects have requirements (taking course EA2 requires course EA1).

- Dependencies are modelled in a graph. Edge (u, v) means "can't do project u without also doing project v."

- A subset A of projects is feasible if the prerequisites of every project in A also belong to A.
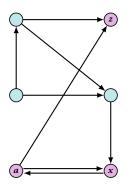
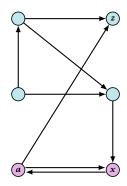**Goal:** Find a feasible set of projects that maximizes the profit.

Project Selection

The prerequisite graph:

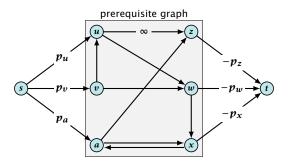- ▶ $\{x, a, z\}$ is a feasible subset.
- ▶ $\{x, a\}$ is infeasible.

Project Selection

Mincut formulation:

- ▶ Edges in the prerequisite graph get infinite capacity.
- ▶ Add edge (s, v) with capacity p_v for nodes v with positive profit.
- ▶ Create edge (v, t) with capacity $-p_v$ for nodes v with negative profit.

prerequisite graph

Theorem 84

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

Proof.

- A is feasible because of capacity infinity edges.

- $\text{cap}(A, V \setminus A) = \displaystyle\sum_{v \in \bar{A} : p_v > 0} p_v + \sum_{v \in A : p_v < 0} (-p_v) = \sum_v p_v - \sum_{v \in A} p_v$

prerequisite graph