Superconvergence effect and adaptive refinements in FEM

Nowadays finite elements method is one of the most powerful computational tools at our hands. So it is not surprising that subject is developing and many important supplements are continuously appearing. Here we consider superconvergence the way to improve accuracy of derivative of FEM solution, and adaptive refinements, way to accuracy of FEM solution itself.

FEM. Main idea

Replacement of the problem in the infinite dimensional space by linear algebraic problem (discretization). Let's consider linear differential operator in the following problem:

$$\begin{cases} L\big(u(\boldsymbol{x})\big) = f(\boldsymbol{x}) \quad u \in \mathcal{U} \subset \mathrm{C}^{\mathrm{q}}_{\mathbb{R}^{\mathrm{n}}} \quad \boldsymbol{x} \in \mathcal{X} \subset \mathbb{R}^{\mathrm{n}} \\ u(\boldsymbol{x})_{\partial\Omega} = \eta(\boldsymbol{x}) \end{cases} \tag{1}$$

Our goal is to discretize (1) and obtain system of linear algebraic equations

$$\mathcal{L}\vec{u} = \vec{f}$$

For this aim one chooses a grid for Ω. It consists (in general) of some closed areas, in, for example, 2D case it is triangles, squares or either curvilinear polygons.

$$\Omega = \bigcup_{i=1}^{m} \Omega_i\,, \quad \Omega_i \cap \Omega_j = 0, \qquad i \neq j$$

On each subdomain Ω_i we can define some finite set of basis functions (polynomials) $\{\phi_{ik}(x)\}_{k=1}^{m}$ with following properties:

i. Normalizing condition $\phi_{ik}\big(\boldsymbol{x}_j\big) = \delta_{kj}$
ii. Set $\{\phi_{ik}(x)\}_{k=1}^{m}$ corresponds to the order of approximation

After this steps we can write down approximate solution as a linear combination of basis functions on each sub domain Ω_i

$$\tilde{u}(\boldsymbol{x}) = \sum_{i=1}^{m}\sum_{k=1}^{n} v_{ik}\phi_{ik}(\boldsymbol{x}) \tag{2}$$

Unknown coefficients in (2) we obtain by weighting residuals method. Residual due to linearity of $L(\cdot)$ can be rewritten as:

$$r(\boldsymbol{x}) = f(\boldsymbol{x}) - L(\tilde{u}(\boldsymbol{x})) = f(\boldsymbol{x}) - \sum_{i,k=1}^{m,n} v_{ik} L(\phi_k(\boldsymbol{x}))$$

Now it's easy to construct the linear algebraic system by weighting $r(\boldsymbol{x})$ with $\phi_j(\boldsymbol{x})$

$$\sum_{i,k=1}^{m,n} v_{ik} \int_{\Omega_j} L(\phi_k(\boldsymbol{x}))\phi_j(\boldsymbol{x})\, d\Omega_j = \int_{\Omega_j} f(\boldsymbol{x})\phi_j(\boldsymbol{x})\, d\Omega_j$$

Adding discretized boundary conditions from (1) one obtains solvable system of linear algebraic equations

Superconvergence

Let the polynomial of highest order in basis $\{\phi_k(x)\}_{k=1}^{m}$ be the polynomial of $\boldsymbol{n}$-th power $\phi_{\tilde{k}}(\boldsymbol{x}) = P_n(\boldsymbol{x})$. For solution (2) error estimation is:

$$e(\tilde{u}) = \|u - \tilde{u}\|_{L^2} \le O(h^{n+1})$$

For derivative of this solution

$$\tilde{u}'(\boldsymbol{x}) = \sum_{i=1}^{m} \sum_{k=1}^{n} v_{ik} \phi'_{ik}(\boldsymbol{x})$$

error estimation is:

$$\phi'_{\tilde{k}}(\boldsymbol{x}) = P_{n-1}(\boldsymbol{x}) \Rightarrow e\left(\widetilde{u'}\right) \le O(h^n)$$

With the help of superconvergence effect it's possible to define derivative of the solution more accurate

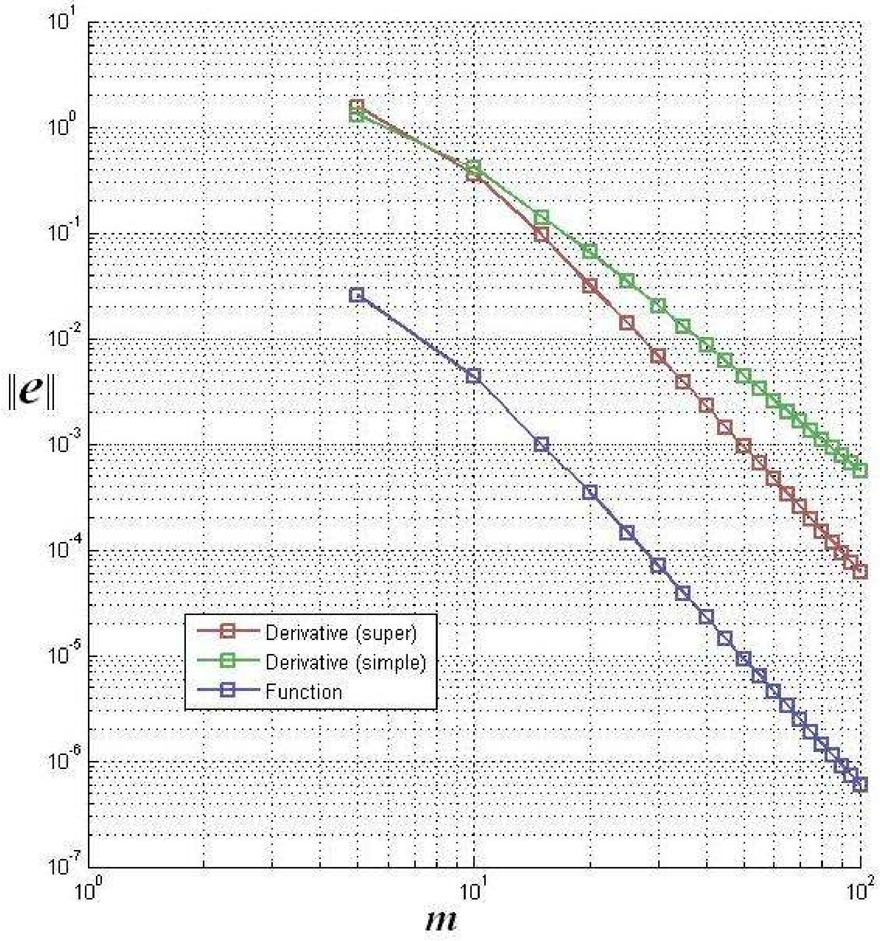

Fig1. Speed of convergence of approximate solution obtained by FEM (blue line), its derivative (green line) and derivative, interpolated in superconvergence points (red line). X-axes number of finite elements (m), Y-axes L2 norm of difference between analytical function and its approximation

We can say that best accuracy of the FEM solution is obtainable for gradients at the Gauss points (roots of Gauss-Legendre polynomial) corresponding, in order, to the polynomial used in the solution. This fact was observed experimentally by Barlow

If once again $\phi_{\tilde{k}}(\boldsymbol{x}) = P_n(\boldsymbol{x})$ then we have ***n*** Gauss points and we can construct only polynomial of order ***n-1***. So we should take some additional points from neighbor finite elements in order to interpolate this data by the polynomial of ***n***-th power. Thus sequence of operations is following

i. Map Gauss-Legendre nodes at the finite element (define points of superconvergence)
ii. Add to the superconvergence points some additional ones from neighbor elements (define value of p-parameter)
iii. Fit the obtained data in the list square sense

In the case of higher dimensional case 2D (3D) we can take finite elements of different forms

- Rectangles (parallelepipeds) – superconvergence points one can obtain from Cartesian product of the corresponding points on the segment (rectangle)
- Triangles (tetrahedrons) – superconvergence points doesn't exist, but still there are some optimal sampling points

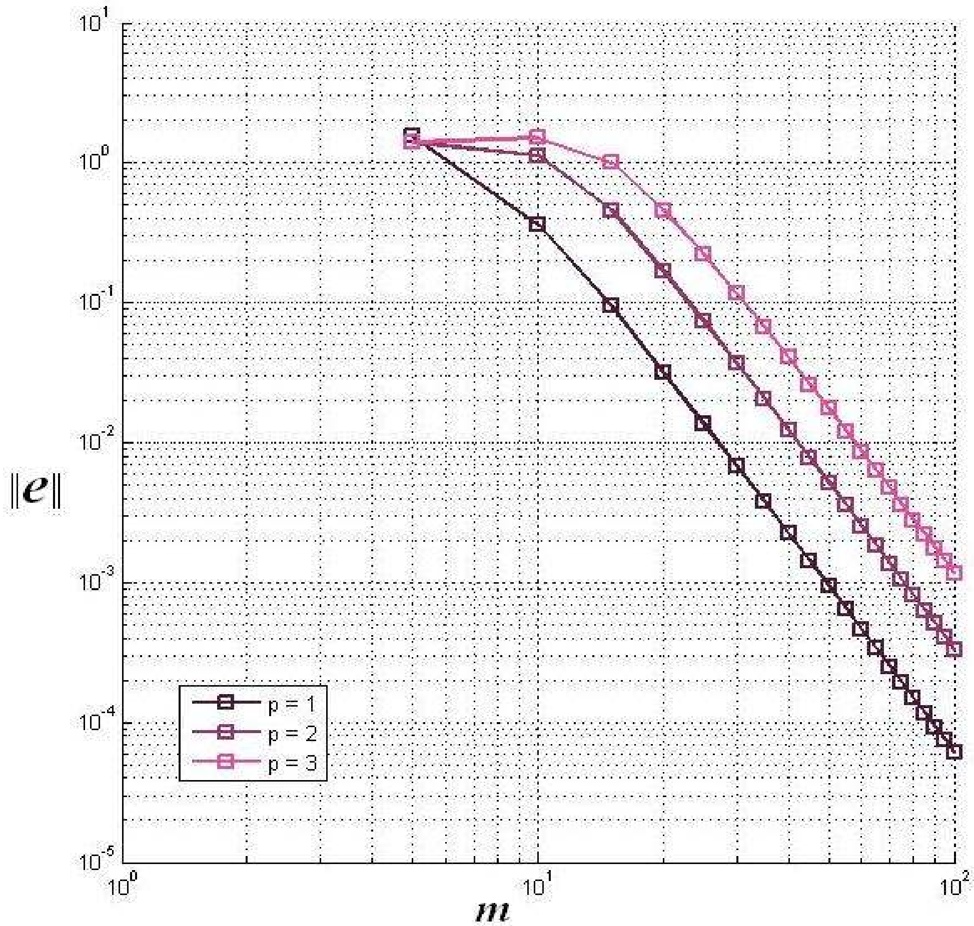

Fig2. Convergence of approximate derivative, interpolated in superconvergence points with different number of outer points. X-axes number of finite elements, Y-axes L2 norm of difference between analytical function and its approximation

Classification of adaptive FE refinement algorithms.

Refinement algorithms help us improve accuracy of solution. Adaptive means dependency on previous results. Various procedures exist for the refinement of finite element solutions. Broadly these fall into two categories:

1) <u>*h*-refinement</u>
   - ∗ Same type of elements
   - ∗ Same type of basis functions

* Elements becomes smaller (larger)

2) <u>*p*-refinement</u>
   * Same size of elements
   * Increases order of approximation functions (locally or throughout whole domain)

Classification of h-refinement algorithms

i. <u>Element subdivision (enrichment)</u>. Here refinement can be conveniently implemented and existing elements, if they show too much error, are simply divided into smaller ones keeping the original elements intact. This method rather simple and widely used.
ii. <u>Mesh regeneration</u>. On the basis of a given solution, a new element size is predicted in all the domain and a totally new mesh is generated. This can be expensive especially in tree dimensions, and it also presents a problem of transferring data from one mesh to another, but still results are generally much superior than in previous case.
iii. <u>Reposition of the nodes</u>. Keeps the total number of nodes constant and adjusts their position to obtain an optimal approximation while this procedure is theoretically of interest it is difficult to use in practice.

References

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