Course "Trees - the ubiquitous structure in computer science and mathematics", JASS'08

Minimum Spanning Trees

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Definition 1

Given a Graph $G = (V, E)$, together with a weight function $w : E \rightarrow \mathbb{R}$. A spanning acyclic Subgraph F with minimum total weight is called a minimum spanning forest (MSF). If F is connected (thus a tree) it is called minimum spanning tree (MST)

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Definition 2

- $F(a, b)$ denotes the path (if exists) in the graph F from node a to node b.
- by default we set $m := |E|$ and $n := |V|$

For simplicity, we assume that all weights in the Graph G are distinct, therefore the MST of G is unique.

Theorem 3 (Cut property)

Let $C = (V_1, V_2)$ be a cut in G, then the lightest edge e crossing the cut belongs to the MST of G.

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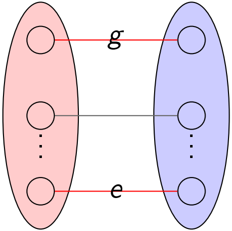
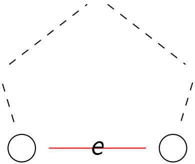
Theorem 4 (Cycle property)

For any cycle C in G the heaviest edge e in C does not belong to the MST of G

Proof.



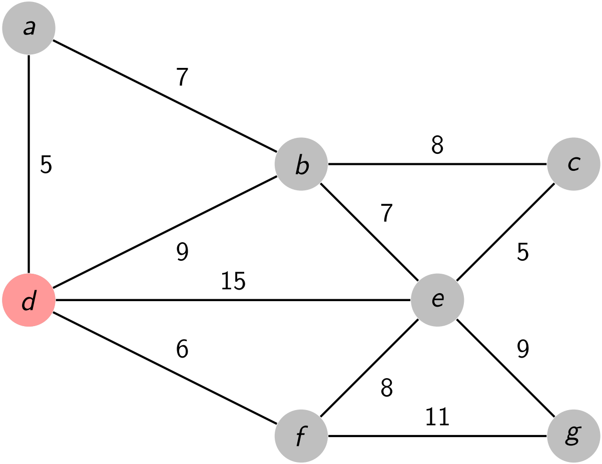


1) Assume $e \notin MST(G)$. Let T be the MST and e be the lightest edge crossing the cut. Then $T \cup e$ yields a circle that includes at least two edges e and g crossing the cut. Exchanging e for g gives a MST with lower weight - contradiction!
2) Assume $e \in MST$. Deleting $e = (v, w)$ splits the Tree in two parts which can be reconnected using an edge from the circle. This edge is lighter than e and so is the resulting spanning tree - contradiction! □

Dijkstra-Jarnik-Prim Algorithm (DJP) Grows a tree T, one edge per step, starting with a tree consisting of one arbitrary vertex. Augment T by choosing an edge incident to T having the least weight. By the cut property this edge belongs to the MST of the Graph.

$V(T) := \{v\};\ E(T) = \emptyset$
<u>for</u> $n-1$ times
  choose lightest edge (x, y) indicent to T
    with $x \in T$ and $y \notin T$
  $V(T) := V(T) \cup \{y\};\ E(T) := E(T) \cup (x, y)$
<u>end</u>

Runtime: $\mathcal{O}(m + n \log n)$ if implemented using Fibonacci-Heaps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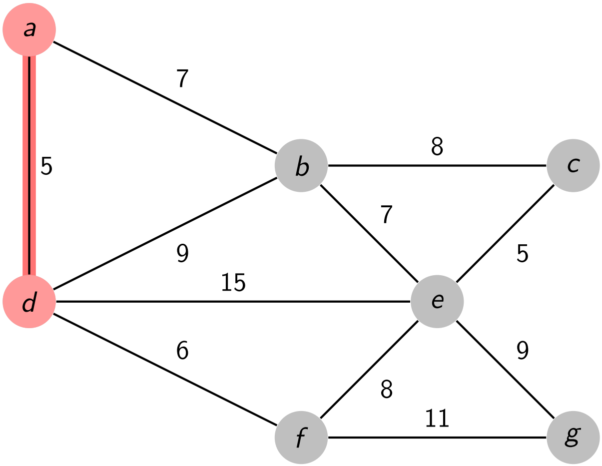
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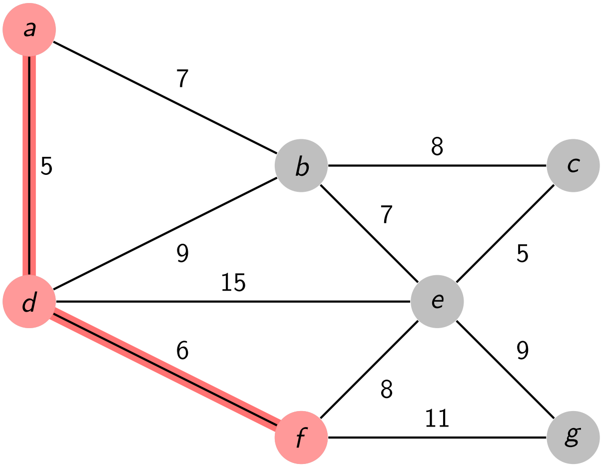
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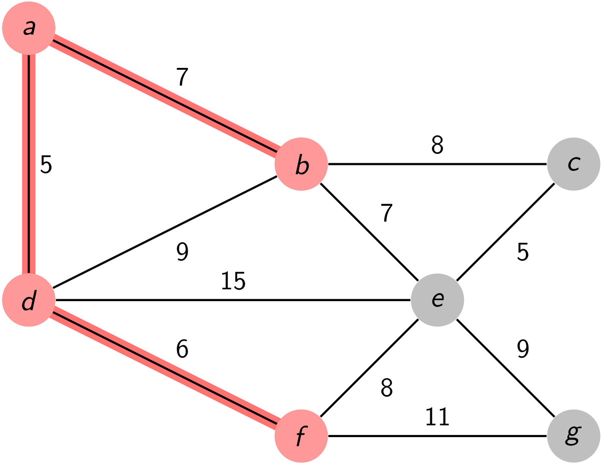
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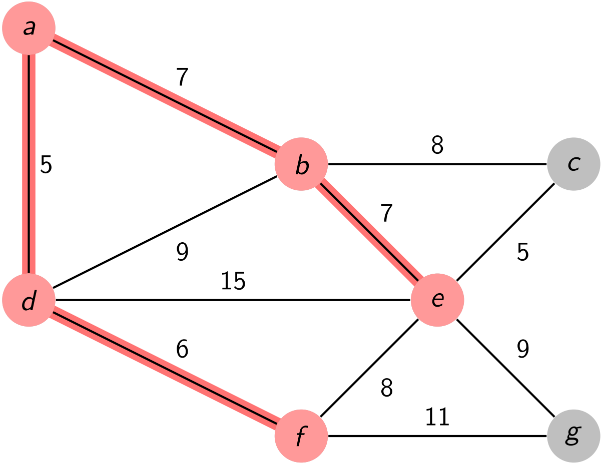
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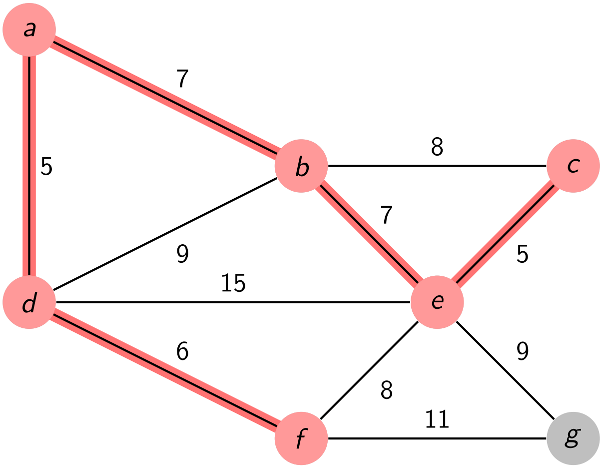
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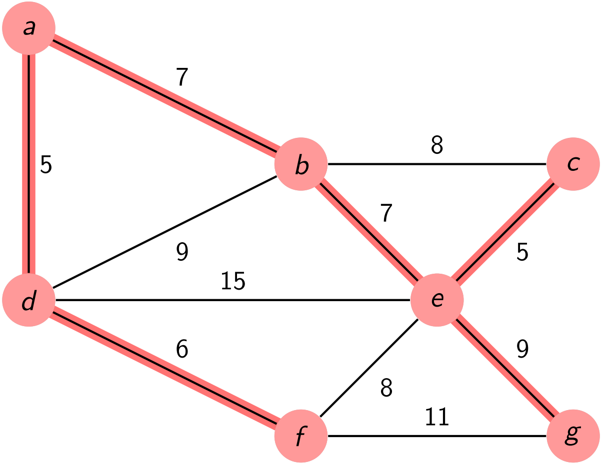
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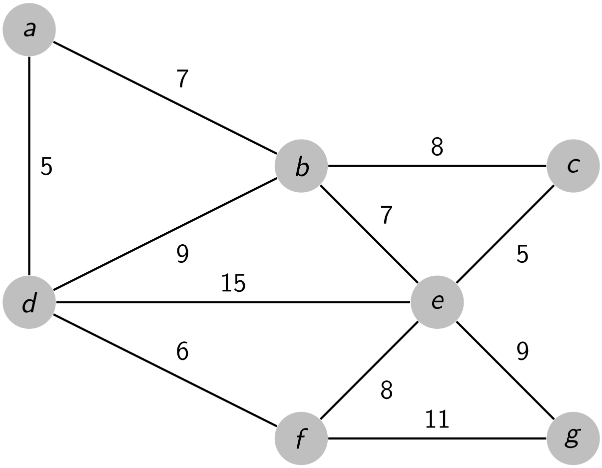
Kruskal's Algorithm:

- Sort all edges according to their weight in ascending order.
- Include edges successively in the MSF if they do not complete a circle.

Correctness follows directly from cut and cycle properties.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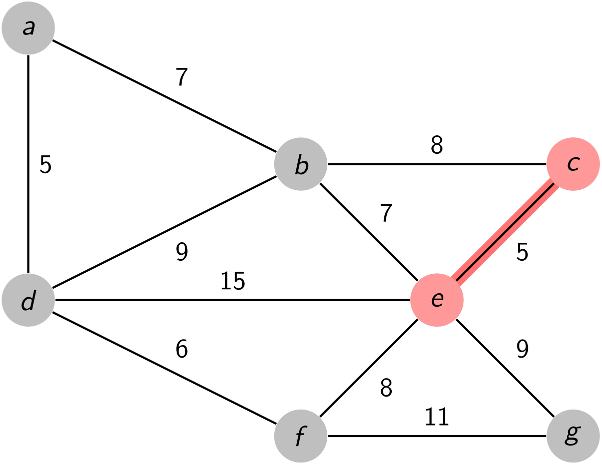
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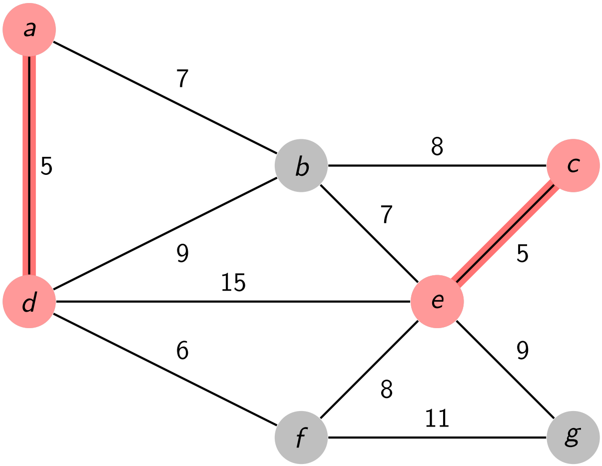
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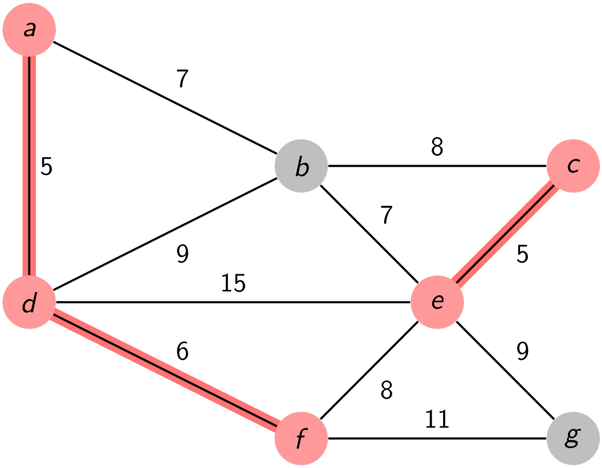
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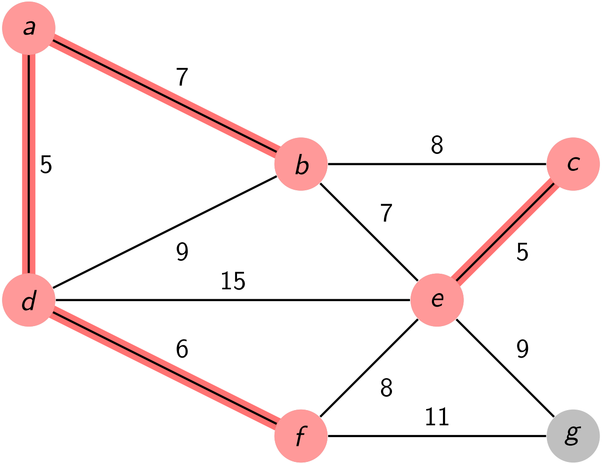
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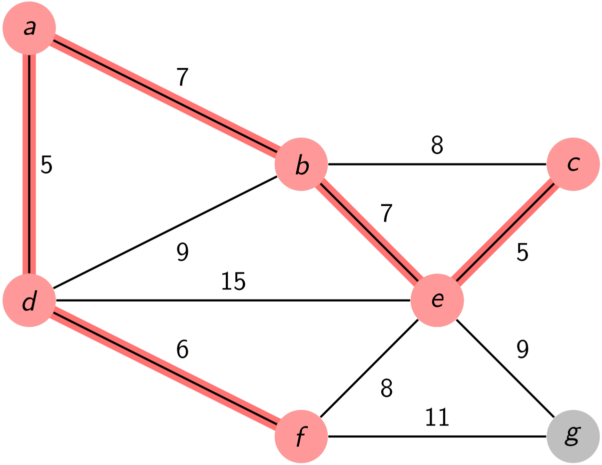
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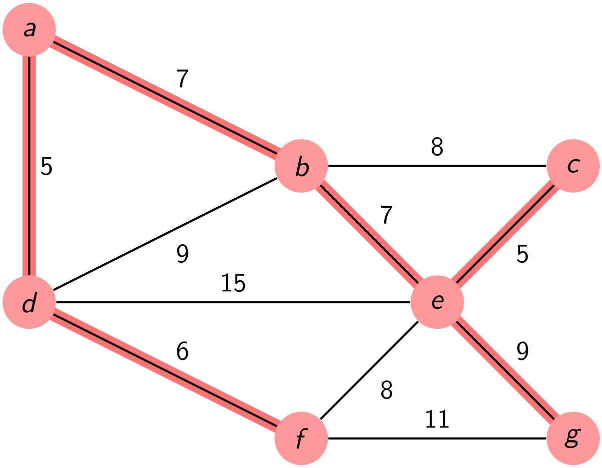
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Borůvka's Algorithm

```
make every vertex a singleton red tree
repeat until there is one red tree
  for each red tree
    select minimum weight edge incident to it
  color all selected edges red
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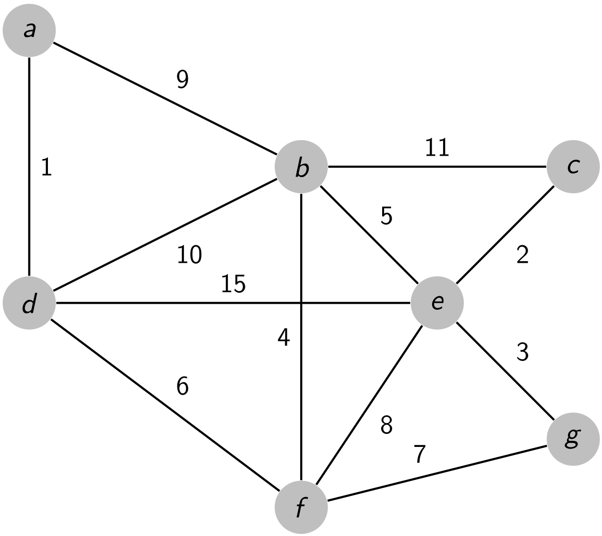
Each inner execution of the loop is called a Borůvka step. Each step reduces the number of vertices by at least 2 and takes $\mathcal{O}(m)$ time, therefore the total running time is in $\mathcal{O}(m \log n)$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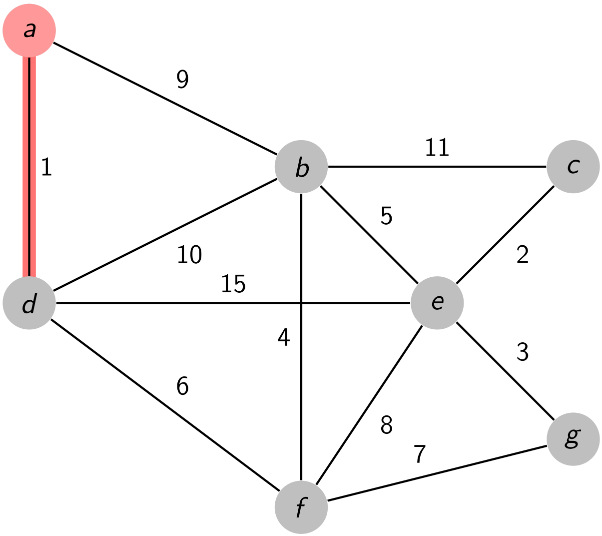
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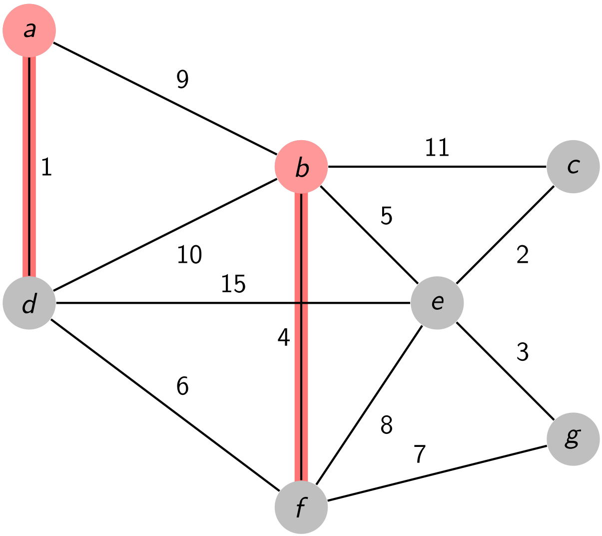
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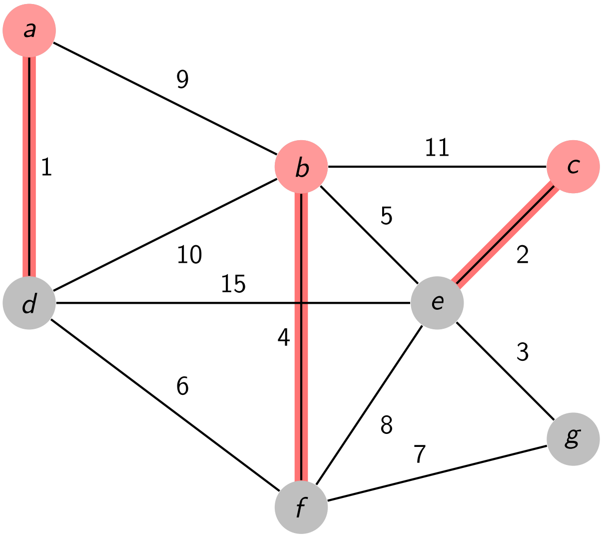
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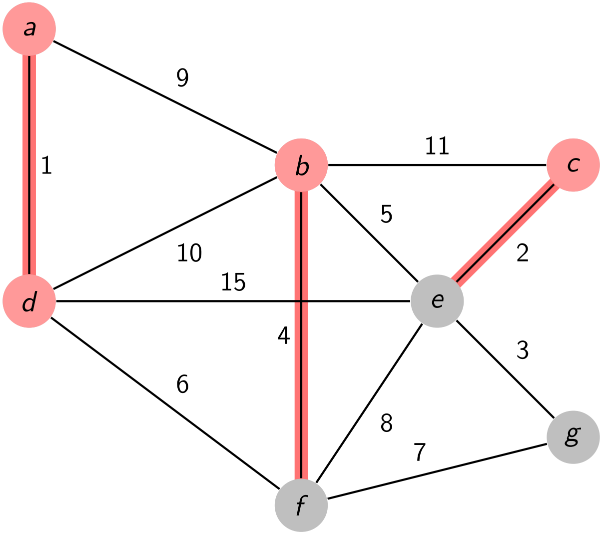
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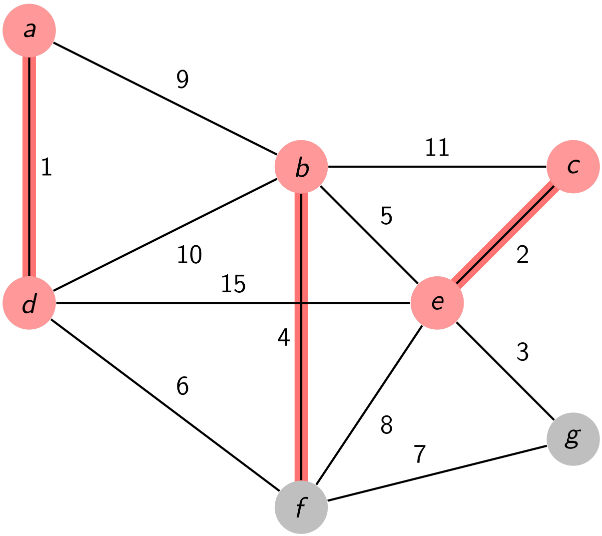
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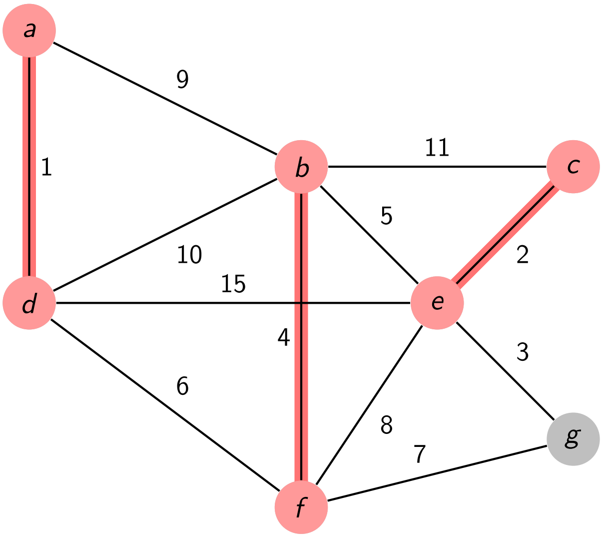
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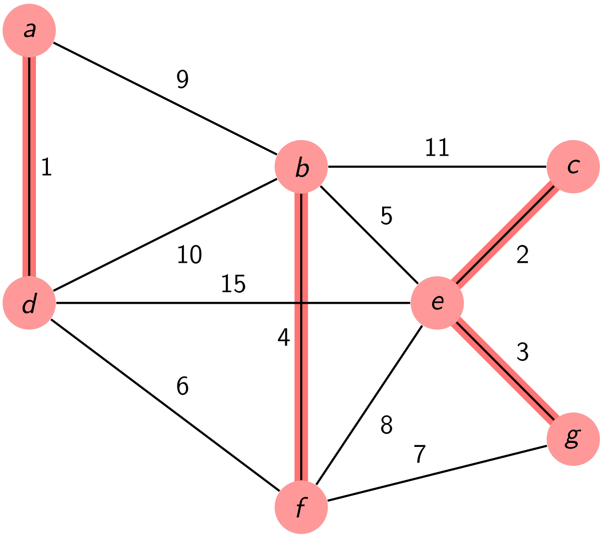
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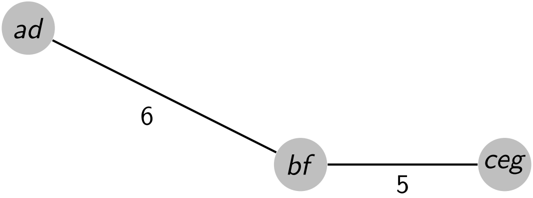
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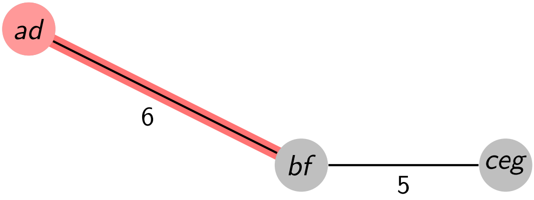
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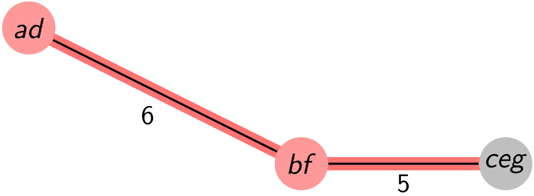
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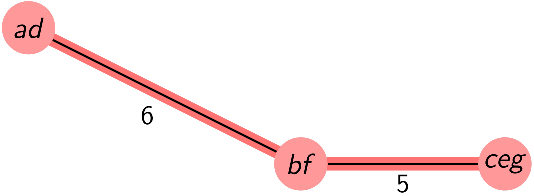
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MST Verification

Theorem 5

A spanning tree is a MST iff the weight of each nontree edge (u, v) is at least the weight of the heaviest edge in the path in the tree between u and v.

Definition 6

Tree path problem: finding the heaviest edges in the paths between certain pairs of nodes (”query paths“).

Solved by Komlòs in 1984; algorithm requires linear comparisons but nonlinear overhead!

King [Kin97] presented a simpler algorithm using Komlòs' algorithm on a full branching tree B which gives linear runtime (on a unit cost RAM)!

Theorem 7

If T is a spanning tree then there is an $\mathcal{O}(n)$ algorithm that constructs a full branching tree B s.t.

- *B has not more than $2n$ nodes*
- *For any pair of nodes x and y in T, the weight of the heaviest edge in $T(x, y)$ equals the weight of the heaviest edge in $B(x, y)$*

Construction of B: We run Borůvka's algorithm on a **tree** T

- for each node v in V we create a leaf $f(v)$ for B
- let $A = \{v \in V| \text{ v contracted into t by Borůvka step}\}$. Add new node $f(t)$ to B add $\left\{(f(a), f(t))|\forall a \in A\right\}$ to the set of edges in B

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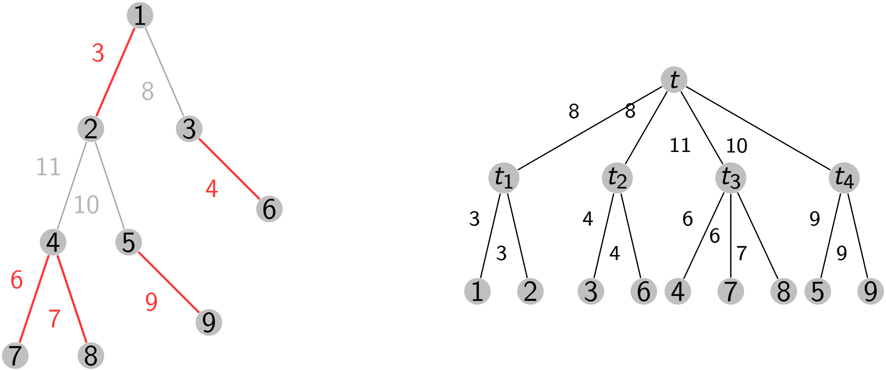
The runtime of a Borůvka step is proportional to the number of "uncolored" edges. This number drops by a factor of 2 after each step (because T is a tree) $\Rightarrow$ runtime is in $\mathcal{O}(n)$.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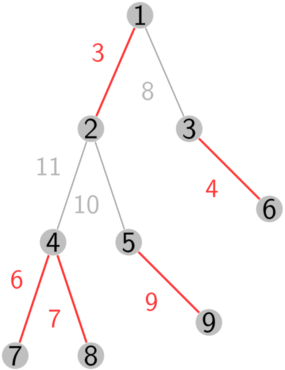
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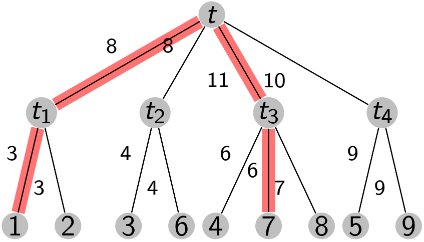
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Komlòs algorithm on full branching trees:

- Goal: Find the heaviest edge between every pair of leaves
- Idea: Break each path in two half-paths, from leaf to the lowest common ancestor. Then find heaviest edge in each half-path
- Finding the heaviest edge in a query path then requieres one additional comparison

Definition 8

- $A(v)$ denotes the set of paths which contain v and are restricted to the "interval" $[root, v]$
- $A(v|a)$ denotes the set of restrictions of each path in $A(v)$ to the interval $[root, a]$
- $H(p)$ denotes the weight of the heaviest edge in a path p

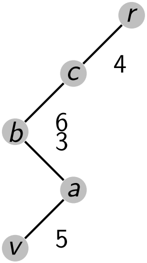
Example:

- $A(v) = \{(v, a), (v, a, b), (v, a, b, c), (v, a, b, c, root)\}$
- $A(v|a) = \{(a, b), (a, b, c), (a, b, c, root)\}$
- $length(s) > length(t) \Rightarrow H(s) \geq H(t)$ for any two paths s, t in $A(v)$ therefore the order of $H(A(v))$ is determined by length of paths.

Algorithm:

- Starting with the root descend the Tree level by level
- At each node v, the heaviest edge in the Set $A(v)$ is determined:
  - Let p be the parent of v and assume we know $H(A(p))$.
  - Then $H(A(v))$ can be found by comparing v, p to each weight in $H(A(p)$ using binary search.

Komlòs showed, that $\sum_{v \in T} \log |A(v)| \in \mathcal{O}(n \log((m+n)/n)) \subset \mathcal{O}(m)$, which is an upper bound on the number of comparisons needed to find the heaviest edge in all half-paths.



$$A(r) = \emptyset$$
$$A(c) = \{(c, r)\}$$
$$A(b) = \{(b, c), (b, c, r)\}$$
$$A(a) = \{(a, b), (a, b, c), (a, b, c, r)\}$$
$$A(v) = \{(v, a), (v, a, b), (v, a, b, c), (v, a, b, c, r)\}$$
$$H(A(c)) = \{4\}$$
$$H(A(b)) = \{6, 6\}$$
$$H(A(a)) = \{3, 6, 6\}$$
$$H(A(v)) = \{5, 5, 6, 6\}$$

Summary Verify that T is a MST

- generate full-branching tree B via Borůvka algorithm applied to T
- precompute the heaviest edge of all half-paths in B
- precompute all lowest common ancestors in B for the leaves x and y that form a non-tree edge (x, y) in T
- for every non-tree edge $e = (x, y)$ in T compare $w(e)$ to heaviest edge in half-paths $B(x, lca)$ and $B(lca, y)$

Remark: the LCA of all pairs can be computed in $\mathcal{O}(m + n)$, therefore the total running time of the algorithm is in $\mathcal{O}(m)$

A Randomized Linear-Time Algorithm for MST[DRK95]

Definition 9

Let G be a weighted graph and F be a forest in G.

- $w_F(x, y)$ denotes the maximum weight of an edge on $F(x, y)$
- An edge (x, y) is called F-heavy if $w(x, y) > w_F(x, y)$ and F-light otherwise

Note:

- All edges of F are F-light
- For any forest F, no F-heavy edge can be in the MSF of G by the cycle property.

Algorithm

Step(1) Apply two Borůvka steps to the graph, reducing the number of vertices by at least a factor of 4

Step(2) In the contracted graph choose a subgraph H by selecting each edge with probability $1/2$. Apply the algorithm **recursively** on H, to get a MSF F of H. Find all the F-heavy edges (both those in H and not in H) and delete them.

Step(3) Apply the algorithm **recursively** to the remaining graph to compute a spanning forest F'. Return the edges contracted in Step(1) together with the edges of F'

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Remarks:

- all edges in $H - F$ are F-Heavy, but there may be more in the rest of G
- only the edges that are in F appear in **both** recursions

Correctness is proved by induction and cycle property (every edge that is deleted in Step(2) cannot be in the MSF)

Analysis:

Step(1) takes time $\mathcal{O}(m)$ (2 Borůvka steps).

Step(2) finding F-heavy edges takes time $\mathcal{O}(m)$ using Komlòs algorithm as described in [Kin97]

so for some constant c we can describe the runtime as:

$$T(n,m) = cm + \underbrace{T(n_2, m_2)}_{\text{recursion in Step(2)}} + \underbrace{T(n_3, m_3)}_{\text{recursion in Step(3)}}$$

- imagine recursion tree: left child of a node is seen as the recursion in Step(2), right child as the recursion in Step(3)

Theorem 10

Wort-case Runtime of the algorithm is in $\mathcal{O}(\min\{n^2, m \log n\})$

Proof.

- consider subproblem in depth d: num. of nodes $\leq n/4^d \Rightarrow$ num. of edges $\leq (n/4^d)^2/2$
- sum over all subproblems: total num. of edges $\leq \sum_{d=0}^{\infty} 2^d \frac{n^2}{2 \cdot 4^{2d}} \leq \frac{n^2}{2} \sum_{d=0}^{\infty} \frac{2^d}{2^{2d}} = n^2$

Proof (continued).

- Parent problem with v vertices. Edges of F are in both subproblems, edges revomed in Step(1) are in none. All other edges are in exactly one subproblem.
- After Step(1) there are $v' \leq v/4$ nodes left in $G \Rightarrow F$ has $\leq v' - 1 \leq v/4$ edges. But at least $v/2$ edges are removed in Step(1) therefore the total number of edges in both subproblems does not increase.
- $\Rightarrow$ Number of edges in all subproblems at depth d in recursion tree is $\leq m \Rightarrow$ as the recursion tree has depth $\mathcal{O}(\log n)$ the runtime is in $\mathcal{O}(m \log n)$

□

Theorem 11

The expected runtime of the algorithm is in $\mathcal{O}(m)$

Proof.

- $X :=$ number of edges in parent problem
- $Y :=$ number of edges in left subproblem

In Step(2) each edge that was not removed in Step(1) is included with probability $1/2 \Rightarrow$
$E[Y|X = k] \leq k/2 \Rightarrow E[Y|X] \leq X/2 \Rightarrow E[Y] \leq E[X]/2$

$$T(n, m) = cm + \underbrace{T(n_2, m_2)}_{\text{recursion in Step(2)}} + \underbrace{T(n_3, m_3)}_{\text{recursion in Step(3)}}$$

$$T(n, m) \leq cm + T(n/4, m/2) + T(n/4, \underbrace{n/2)}_{\text{by Lemma 12}}$$

$$\Rightarrow T(n, m) \in \mathcal{O}(m + n) \subset \mathcal{O}(m)$$

Lemma 12

Let H be a subgraph of G obtained by including each edge independently with probability p and let F be the MSF of H. Then the expected number of F-light edges in G is $\leq n/p$.

Proof.

Using the mean of the negative binomial distribution. see [DRK95] □

An Optimal MST Algorithm:

- Pettie and Ramachandran[SP01], asymptotically optimal on a pointer machine
- Uses precomputed optimal decision trees (unknown depth $\Rightarrow$ exact runtime not known!)
- Fredman and Tarjan [MLF87] showed how to compute the MST in time $\mathcal{O}(m\beta(n, m))$ with $\beta(n, m) = \min\{i | \log^{(i)} n < n/m\}$

$\Rightarrow$ For graphs with density $\Omega(\log^{(3)} n)$ this yields a linear-time algorithm $\rightsquigarrow$ DenseCase algorithm.

Central datastructure used is the Soft Heap [Cha00]

- approximate priority queue with fixed error rate
- upports all heap operations (Insert, FindMin, Delete, Union) in constant armotized time except for Insert which takes time $\mathcal{O}(\log(\frac{1}{\epsilon})$
- Items are grouped together sharing the same key. Items can adopt larger keys from other items corrupting the item.

This is shown in [Cha00]:

Lemma 13

For any $0 < \epsilon \leq 1/2$, a soft heap with error rate ϵ supports each operation in constant amortized time, except for insert, which takes $\mathcal{O}(\log(\frac{1}{\epsilon}))$ time. The data structure never contains more than ϵn corrupted items at any given time.

Lemma 14 (DJP Lemma)

Let T be a tree formed after some number of steps of the DJP algorithm. Let e and f be two arbitrary edges with exactly one endpoint in T and let g be the maximum weight edge on the path from e to f in T. Then g cannot be heavier than both e and f.

Lemma 14 (DJP Lemma)

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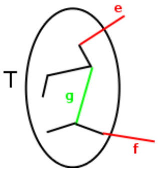
Proof.



Let $\mathcal{P}$ be the path connecting e and f, assume the contrary, that g is the heaviest edge in $\mathcal{P} \cup \{e, f\}$. At the moment g is selected by DJP there are two edges eligible one of which is g. If the other edge is in $\mathcal{P}$ then it must be lighter than g. If it is either e or f then by the assumption it must be lighter than g. In both cases g could not be chosen next by DJP so we have a contradiction. □

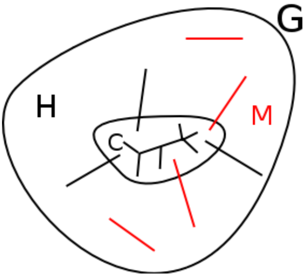
Definition 15

Let F be a subgraph of G. $G \setminus F$ denotes the graph that results from G by contracting all connected components formed by F.

Definition 16

Let M and C be Subgraphs of G.

- $G \Uparrow M$ the graph obtained from G when raising the weight of every edge in M by an arbitrary amount (these edges are *corrupted*)
- M_C is the set of edges in M with exactly one endpoint in C
- C is said to be DJP-contractable if after some steps of the DJP algorithm with start in C the resulting tree is a MST of C


G
H
M
C

Lemma 17 (Contraction lemma)

Let M be a set of edges in a graph G. If C is a subgraph of G that is DJP-contractable w.r.t. $G \Uparrow M$, then

$$MSF(G) \subset MSF(C) \cup MSF(G \setminus C - M_C) \cup M_C$$

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Proof [SP01].

We prove $MSF(G)^C \supset \underbrace{MSF(C)^C}_{(1)} \cap MSF(G \setminus C - M_C)^C \cap M_C^C$

where A^C denotes the complement of the set A (concerning the edges, so $MSF(C)^C = C - MSF(C)$)

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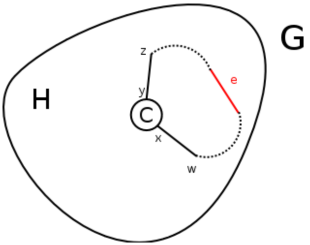
(1) Every edge in C that is not in $MSF(C)$ is the heaviest edge on a cycle in C (because C has a MST). This cycle exists in G as well, so this edge is also not in the MSF of G.

Proof cont.

It remains to show that $MSF(G)^C \supset MSF(G \setminus C - M_C)^C \cap M_C^C$.
Set $H := G \setminus C - M_C$. Then we are left with

$$MSF(G)^C \supset H - MSF(H) \cap \underbrace{G \setminus C - M_C}_{=H} = H - MSF(H)$$

Let $e \in H - MSF(H)$, then e is the heaviest edge on some cycle χ in H.



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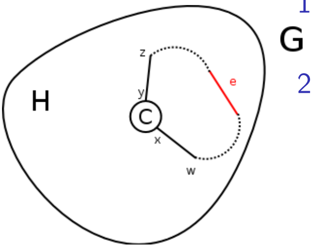
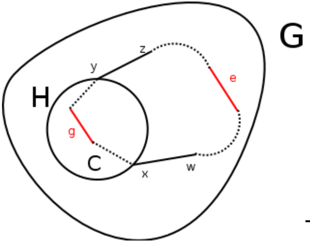


1) If χ does not involve the super-node C then it exists in G as well and $e \notin MSF(G)$.
2) Otherwise χ includes a path $\mathcal{P} = (x, w, \ldots, z, y)$ in H with $x, y \in C$. Since H includes no corrupted edges with one endpoint in C, the G-weight of the end edges (x, w) and (z, y) is the same as their $(G \Uparrow M)$-weight.

Proof cont.

Let T be the spanning tree of $C \Uparrow M$ that was found by the DJP algorithm, $\mathcal{Q}$ be the path in T connecting x and y, and g be the heaviest edge in $\mathcal{Q}$.



Then $\mathcal{P} \cup \mathcal{Q}$ forms a circle with e being heavier than both (x, w) and (y, z). By the DJP-Lemma 14 The heavier of these both edges is heavier than the $G \Uparrow M$-weight of g which is an upper bound on the G-weigths of all edges in $\mathcal{Q}$. So w.r.t. G-weights, e is the heaviest edge on the cycle $\mathcal{P} \cup \mathcal{Q}$ and thus cannot be in $MSF(G)$ □

Corollary 18

by applying Lemma 17 i times we get

$$MSF(G) \subset \bigcup_{j=1}^{i} MSF(C_j) \cup MSF\left(G \setminus \bigcup_{j=1}^{i} C_j - \bigcup_{j=1}^{i} M_{C_j}\right) \cup \bigcup_{j=1}^{i} M_{C_j}$$

Overview of the optimal algorithm:

1) find DJP-contractable subgraphs $C_1, C_2, \ldots, C_k$ with their associated sets $M = \bigcup_i M_{C_i}$, where M_{C_i} consists of corrupted edges with exactly one endpoint in C.
2) Find MSF F_i of each C_i by using precomputed decision trees for edge weight comparisons. Also find the MSF F_0 of the contracted graph $G \setminus (\bigcup_i C_i) - \bigcup_i M_{C_i}$. By Lemma 17 the MSF of G is contained within $F_0 \cup \bigcup_i (F_i \cup M_{C_i})$.
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Note

- in Step 1) we make sure that each C_i is extremely small ($< \log^{(3)} n$ vertices) so we can apply the decision trees in Step2)
- until Step 3) no edges of the MSF of G have been identified - we only have discarded lots of edges.
- F_0 in Step can be found by the DenseCase algorithm

This procedure finds the DJP-contractable subgraphs and the set M

Partition $(G,\ maxsize,\ \epsilon)$ returns $M, \mathcal{C}$

```
All vertices are initially ''live''
M := ∅;  i := 0
While there is a live vertex
  i := i + 1
  Let V_i := {v} where v is any live vertex
  Create a Soft Heap consisting of v's edges
  While all vertices in V_i are live and |V_i| < maxsize
    Repeat
      delete min-weight edge (x, y) from Soft Heap
    Until y ∉ V_i
    V_i := V_i ∪ y
    If y is live then insert each of y's edges into the Soft Heap
  Set all vertices in V_i to be dead
  Let M_{V_i} be the corrupted edges with one endpoint in V_i
  M := M ∪ M_{V_i}      G := G − M_{V_i}
  Dismantle the Soft Heap
Let C := {C_1, ..., C_i} where C_k is the subgraph of G induced by V_k
Return M, C
```

- We partition the Graph into DJP contractable components that are very small i.e. have less than $\log^{(3)} n$ vertices.
- The growing of a component stops if it has reached its maximum size, or it attaches to an existing component with $\geq \log^{(3)} n$ vertices
- Then we delete all corrupted edges M_c and contract all remaining connected components into single vertices
- As each connected component consists of $\geq \log^{(3)} n$ vertices the resulting graph has $\leq n / \log^{(3)} n$ vertices and we can apply the DenseCase algorithm to the remaining graph

OptimalMSF(G)

If $E(G) = \emptyset$ then Return($\emptyset$)
$r := \log^{(3)} |V(G)|$
$M, \mathcal{C} :=$ Partition(G, r, ϵ)
$\mathcal{F} :=$ DecisionTrees($\mathcal{C}$)
Let $k := |C|$, let $\mathcal{C} = \{C_1, \ldots, C_k\}$ and $\mathcal{F} = \{F_1, \ldots, F_k\}$
$G_a := G \setminus (F_1 \cup \cdots \cup F_k) - M$
$F_0 := DenseCase(G_a)$
$G_b := F_0 \cup F_1 \cup \cdots \cup F_k \cup M$
$F', G_c :=$ Boruvka2(G_b)
$F :=$ OptimalMSF(G_c)
Return $(F \cup F')$

Analysis: Apart from recursive calls the computation is clearly linear. Partition takes $\mathcal{O}(m \log(1/\epsilon))$ time and because of the reductions in vertices DenseCase also takes linear time. For $\epsilon = \frac{1}{8}$ the number of edges passed to the recursive calls is $\leq m/4 + n/4 \leq m/2$ which gives a geometric reduction in the number of edges. The lower bound for any MSF algorithm is $\mathcal{O}(m)$, so the only bottleneck, if any, must lie in the decision trees, which are optimal by construction. One can quite easily show

$$T(m, n) \in \mathcal{O}(\mathcal{T}^*(m, n))$$

if T is the runtime of our algorithm and T^* is the optimal number of comparisions needed for determining the MSF of an arbitrary graph.

Summary

- We can verify a MST in linear time on a RAM with wordsize *logn*
- There is an randomized algorithm that runs in expected linear time and w.h.p. in "real" linear time
- The MST can be computed optimally on a pointer machine - but we do not know the worst case runtime

Open problems:

- Is there a linear time algorithm that runs on pointer machines?
- Is there an optimal algorithm that does not use precomputed decision trees?
- Can we find good parallel algorithms for the MST problem?

B. Chazelle.
The soft heap: An approximate priority queue with optimal error rate.
Journal of the ACM, 47(6):1012–1027, 2000.

R. E. Tarjan D. R. Karger, P. N. Klein.
A Randomized linear-time algorithm to find Minimum Spanning Trees.
Journal of the ACM, 42(2):321–328, 1995.

V. King.
A simpler Minimum Spanning Tree Verification algorithm.
Algorithmica, 18(2):263–270, 1997.

R. E. Tarjan M. L. Fredman.
Fibonacci heaps and their uses in improved network optimization algorithms.
Journal of the ACM, 34(3):596–615, 1987.

V. Ramachandran S. Pettie.
An Optimal Minimum Spanning Tree algorithm.
Technical report, Department of Computer Sciences, The University of Texas at Austin, 2001.
URL:
http://www.cs.utexas.edu/users/vlr/papers/optmsf.pdf.