Tutte Polinomial

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  - $k(G)$ is number of connectivity components.
  - $H \subset G$ if H is subgraph of G.

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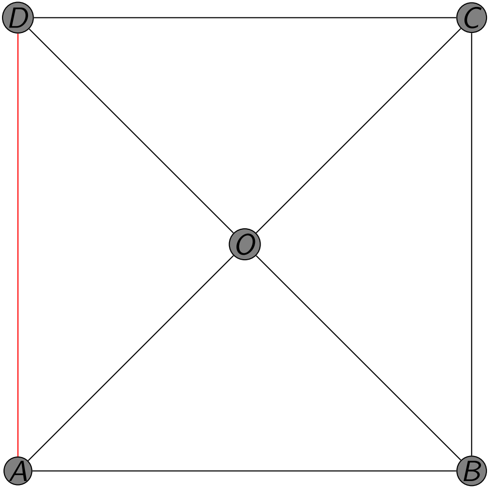
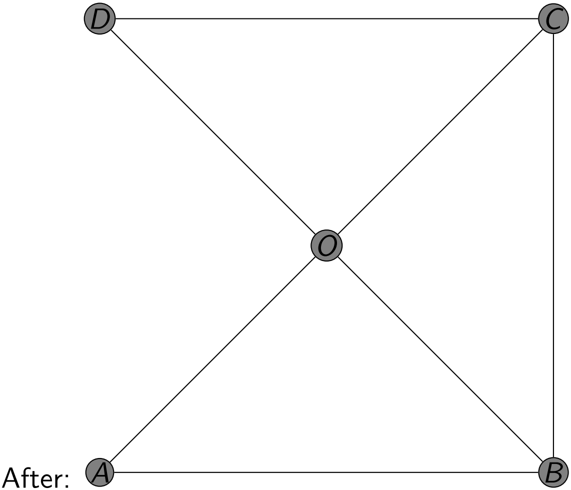
- contraction.

Deletion

Before: 

Deletion



Contraction

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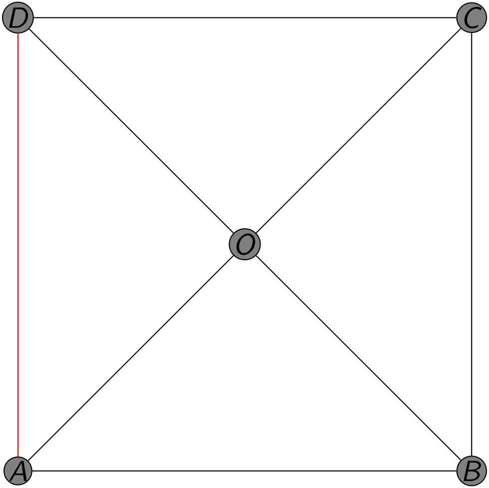

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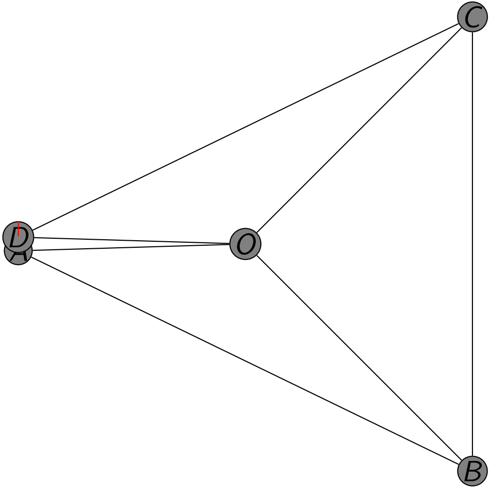

After:

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  If e is incident with u and v then in G/e vertices u and v are replaced by single vertex $w = (uv)$ and each element $f \in E - \{e\}$ that is incident with either u or v is replaced be an edge or loop incident with w.

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So C_G is function $\mathbb{N}_0 \to \mathbb{N}_0$

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Note: 0^0 is equal to 1.

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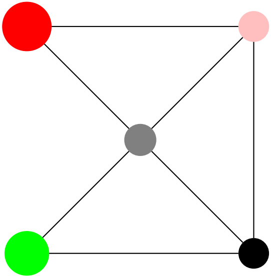
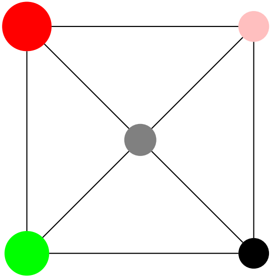
Let $e = (v_1, v_2)$ there two types of coloring G in s colors: in which v_1 and v_2 have different colors and in which they have the same. It's obvious that there are $C(G,s)$ colorings first type and $C(G/e,s)$ second.

Proof's illustration





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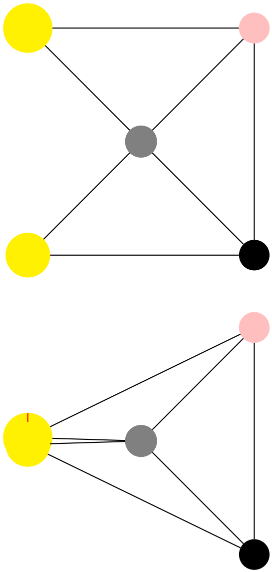

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It implies that $C(G, s)$ is polynomial in s with integer coefficients.

Probability model

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What is probability of graph saving connected?

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$$R(G,p) = \sum_{\substack{H \subset G \\ V(H)=V(G) \\ k(H)=k(G)}} P_{G,p}(H)\mathrm{Connect}(H)$$

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Like previous, $R(G, p)$ is polynomial with integer coefficients.

Spanning trees

Let $B(G)$ is number of G's spanning trees.

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Important idea

It is interesting that $C(G), R(G), B(G)$ and many others graph invariants (if they satisfy **contraction-deletion** relationships) can be expressed from one more general graph invariant, named Tutte polynomial.

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It is interesting that $C(G), R(G), B(G)$ and many others graph invariants (if they satisfy **contraction-deletion** relationships) can be expressed from one more general graph invariant, named Tutte polynomial.
There are o lot of way's to define Tutte polynomial and we will try some of them.

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Of course that definition needs in existence proof.

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$$(-1)^{v(G)+k(G)} s^{k(G)} T_G(1-s,0) =$$
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$$A(G) = T(G, 2, 0)$$

Definition 2: *Tutte polynomial* $T_G(x, y)$ by definition is equal to

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Proof: Can be an exercise.

Special values

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$\#\{H$ is spanning tree$\}$.

Special values

So

$$T_G(1,1) = \sum_{\substack{H \subset G \\ V(H)=V(G)}} 0^{k(H)-k(G)} 0^{e(H)-v(G)+k(H)}$$

is equal to number of spanning trees.

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$$=$$

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- Try to do it with Brooks theorem
- Try to find sum of coefficients Tutte polynomial for K_n
  Note: it is value in $(1, 1)$ equals to number of spanning trees equals to n^{n-2} as we know.

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We have seen that all over the word can be expressed from Tutte polynomial, so it save a lot of information about graph.
And, for example, chromatic polynomial can lose almost all information about graph if it has a loop.

No magic

We have seen that all over the word can be expressed from Tutte polynomial, so it save a lot of information about graph.
And, for example, chromatic polynomial can lose almost all information about graph if it has a loop.
It can be explained very easy.

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And U can be expressed from T!

Universal polynomial's construction

$$U(G) = \alpha^{k(G)}\sigma^{e(G)-v(G)+k(G)}\tau^{v(G)-k(G)}T(G, \frac{\alpha x}{\tau}, \frac{y}{\sigma})$$

Many formulae from that presentation can be obtained from it.

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$$C(G, s) = U(G, 1, 0, s, 1 - 1)$$

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In second summand we can contract e

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It can be an exercise - to check that it statement satisfies properties of Tutte polynomial.

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Let σ is system's state; $\sigma(e)$ is equal to one if vertices, incident e have same states and 0 in other cases.
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$$\frac{\exp(-\frac{1}{kT}\Pi(\sigma_0))}{\sum_{\sigma} \exp(-\frac{1}{kT}\Pi(\sigma))}$$

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So denominator is equal to $Z(G, q, v)$