# 1 Option pricing with sparse grid quadrature

## 1.1 Option definition and its pricing

Option is a financial contract which allows its buyer to buy (call) or sell (put) a specified asset at a set price on or before a future expiry date. The buyer can, but does not have to exercise the contract - the seller on the other hand is always obliged to fulfill buyers wish. Buyer of the option has to pay a certain fee for it. There are various kinds of options and they differ by the number of underlying assets and the execution time. As there are call and put options, and we can either buy (take a long position) or sell (take a short position) any of them, we have four basic positions.

The main use of options is to avoid risk. If we buy/sell both the option and the underlying asset, we may reduce the possible loss and risk, but by doing so we also reduce the possible gain. One can also try to achieve a very high income by buying the option alone - the rewards can be high, but the risk is huge as well, as the payoff for options is given by equations:

$$Payoff(put) = max(K - S, 0)$$
$$Payoff(call) = max(S - K, 0)$$

so we might just as well loose our whole invested money.

### 1.1.1 European option pricing example

In order to find the fair price of an option we have to construct a risk less, self-financing portfolio. Let us assume, that a given stock is worth \$20 today, and in three months time it will be worth either \$18 or \$22. If we want to price a call option with strike price K=\$21, we set a portfolio by buying a number of stocks, $\delta$, and by selling one call option. We finance our portfolio by taking a loan at a rate of $12\%$ yearly, calculated with a continous compound rate. After three months the portfolio should be worth enough money to cover the costs of loan and eventual exercise of the option by the buyer.

- Value of portfolio equal in both cases:
  $\$22 \times \delta - \$1 = \$18 \times \delta - \$0$

- Number of stocks that we have to buy:
  $\delta = 0.25$

- Value of portfolio after three months:
  $\$18 \times 0.25 = \$4.50$

- Current value of portfolio:
  $\$4.50 \times e^{-.12\times 0.25} = \$4.367$

- Current option value:
  $\$20 \times 0.25 - V = \$4.367$
  $V = \$0.633$

### 1.1.2 Option pricing

We model the relative price movement of a stock as a geometric Brownian motion, i.e. $\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma(t)dW(t)$. This is a stochastic differential equation and it can be solved in two ways - either by transforming it into a partial differential equation or by an expectation method using martingale theory. We will use the latter method.

We want to construct a measure under which $\frac{S(t)}{N(t)}, \frac{V(t)}{N(t)}$ are martingales, i.e. $\frac{V(t)}{N(t)} = E(\frac{V(v)}{N(v)}), \forall 0 < t < v < \infty$. If we will calculate the distribution of $S(t)$ under this measure we will be able to solve equation $V(0) = N(0)E^{Q^N}(\frac{V(T)}{N(T)})$. Solution to our problem is given by a new drift $\mu^{Q^N} = r$. A different choice of Numeraire would yield different result, thus we always choose one that is the easiest to calculate. Our final solution is $V(0) = exp(-\bar{r}T)\int_{\infty}^{\infty} max(exp(y) - K, 0)\frac{1}{\sigma\sqrt{T}}\phi(\frac{y-\overline{\mu}}{\sigma\sqrt{T}})$ with phi being a normal distribution $\phi(x) = \frac{1}{\sqrt{2\pi}}exp(\frac{-x^2}{2})$.

If the option would be more complex, for example based on many underlying assets we would receive an integral of many more variables. Numerical integration might require calculation of even few hundred dimensions.

## 1.2 Quadrature

Numerical quadrature is a process of evaluating an integral of some function. In case of one dimensional problems we speak of univariate quadrature and in case of multidimensional problems we obtain multivariate quadrature.

### 1.2.1 Univariate quadrature

Almost all univariate quadrature methods approximate integral of the function by a sum of weighted function values taken at different points $If = \int_{-1}^{1} f(x)dx \approx Qf = \sum_{k=1}^{n} w_k f(x_k)$. Main methods can be divided into three groups:

1. Newton-Cotes with even distances between points and naturally hierarchical, for example Trapezoidal rule - $\int_a^b f(x)dx \approx (b-a)f(\frac{a+b}{2})$.

2. Clenshaw-Curtis uses Chebyshev polynomials, hierarchical.

3. Gauss uses polynomials, the most accurate but not hierarchical, although there are hierarchical variants.

Hierarchical property of a univariate method is very important, as it allows to reuse previous function values computations.

### 1.2.2 Multivariate quadrature

There are four main multivariate quadrature methods:

1. Product of univariate quadrature methods on a full grid - accurate, but extremely inefficient in higher dimensions.

2. Monte Carlo - unaffected by problem dimensionality, but slowly convergent. A random number sequence is used to choose the evaluation points.

3. Quasi Monte Carlo - a more effective version of Monte Carlo, but the evaluation points are chosen not at random but following a low discrepancy sequence in order to cover the integral space as quickly as possible.

4. Sparse grid - uses an optimal grid structure. Allows accuracy similar to full grid methods but with a much lower cost.

## 1.3 Sparse grids

### 1.3.1 Hierarchical basis

In order to create a sparse grid we need to define a hierarchical basis. As the basis function we will use

$$\phi(x) = \begin{cases} 1 - |x| & x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$$

, a simple hat function. Distance between points will be $h_n = 2^{-n}$ and as we operate on interval $[0, 1]$ the grid points are given by $x_{n,i} = ih_n$. Finally with the local basis functions defined as $\phi_{n,i}(x) = \phi(\frac{x - x_{n,i}}{h_n})$ and $0 \leq i \leq 2^{-n}, i\, mod\, 2 = 1$ we have obtained a hierarchical basis with regard to $n$.

In a similar way we can construct a multidimensional hierarchy with the level $n$ defined as a sum of levels in every dimension. When for the two dimensional example we will calculate the costs and gains for each level, we will be able to find the optimal choice of grid levels.

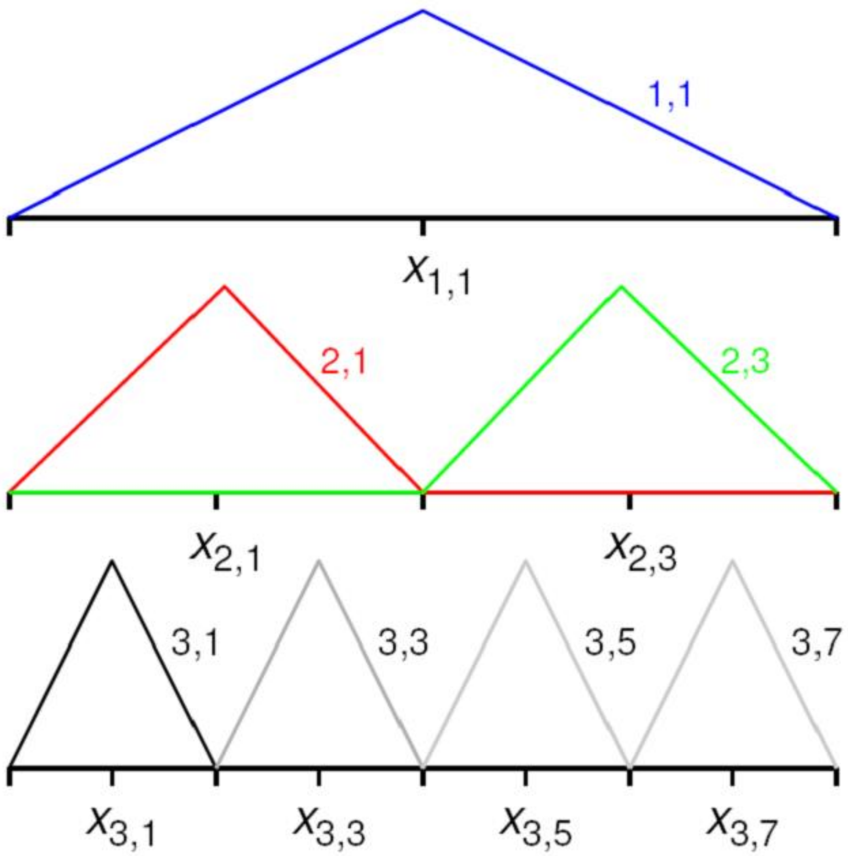


Figure 1.1: One dimensional hierarchical basis.

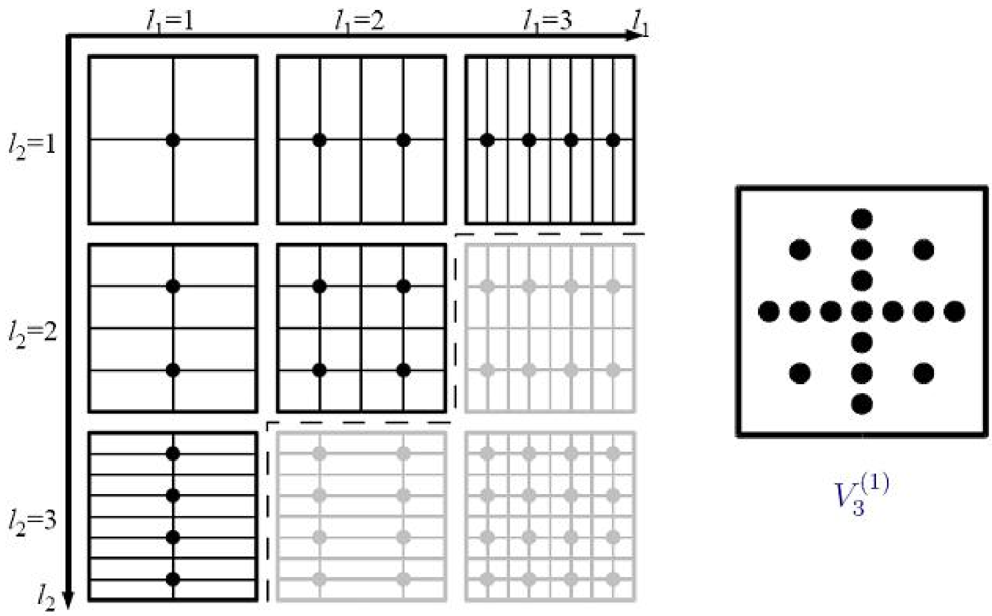


Figure 1.2: Two dimensional hierarchical increments.

### 1.3.2 Smolyak construction

Smolyak proposed a different definition of a sparse grid. His construction has found a widespread use because it is is based on combining standard full grids by addition/subtraction. Such approach allows easy implementation and straightforward parallelisation.

We define a tensor product of two quadrature rules $Q^a f = \sum_{i=1}^{n_1} w_{1,i} f(x_{1,i}), Q^b f = \sum_{i=1}^{n_2} w_{2,i} f(x_{2,i})$ as $(Q^a \otimes Q^b) f = \sum_{j=1}^{n_1} w_{1,j} (\sum_{i=1}^{n_2} w_{2,i} f(x_{1,j}, x_{2,i}))$. In such situation the multivariate quadrature rule takes the form of $Q_l^{(d)} f = \sum_{\|i\| \le l+d-1} (\Delta_{i_1}^{(1)} \otimes ... \otimes \Delta_{i_d}^{(1)}) f$ with the increment $\Delta_i^{(1)} = Q_i^{(1)} - Q_{i-1}^{(1)}$.

## 1.4 Literature


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2. Slides to lecture Scientific Computing 2 Prof. Bungartz, TUM
3. An Introduction to Computational Finance Without Agonizing Pain - Peter Forsyth
4. Mathematical Finance Christian Fries
5. PDE methods for Pricing Derivative Securities - Diane Wilcox