Course "Polynomials: Their Power and How to Use Them", JASS'07

Computing with polynomials: Hensel constructions

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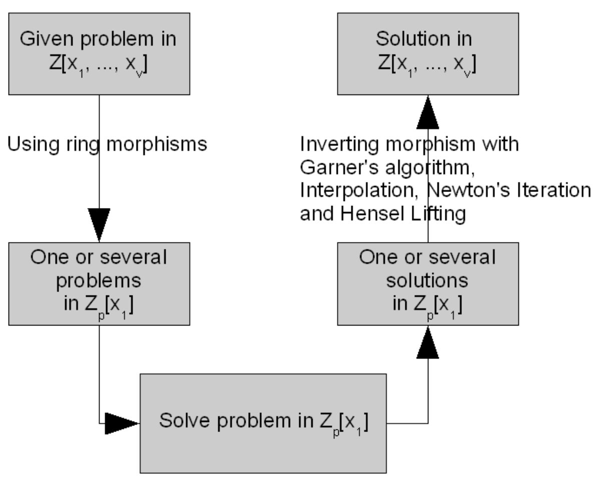
Abstract

To solve GCD calculations and factorization of polynomials in computer algebra systems, they are reduced to problems of simpler domains, from multivariate to univariate domains or from integer numbers to modular rings. Constructing the solution to the original problem from the solution of the simpler problem is called Hensel lifting. We present the necessary background and the idea of this lifting.

1 General background

Motivation and overview



To solve GCD calculations and factorization of multivariate polynomials efficiently, the given problems are projected to one or multiple simpler domains, namely $\mathbb{Z}_p[x_1]$, with ring homomorphisms.
This is abstractly shown in the first figure.
Solving the problem in the simpler domain gives one or multiple solutions for $\mathbb{Z}_p[x_1]$ and these can be lifted to the original domain of the problem,

namely $\mathbb{Z}[x_1, \ldots, x_v]$. While applying the ring homomorphisms is a computationally easy task, inverting the homomorphism is computationally and algorithmically very hard. But because solving the problem in the original domain can be almost unfeasible, the long route by homomorphism methods can decrease the computational cost in many cases.
We will begin with defining ring homomorphisms and especially consider two special homomorphisms, the modular and evaluation homomorphism.

Definition 1 (ring homomorphism)**.** Let R and R' be two rings. Then a mapping $\theta : R \to R'$ is called a ring homomorphism if

1. $\theta(a + b) = \theta(a) + \theta(b)$ for all $a, b \in R$
2. $\theta(ab) = \theta(a)\theta(b)$ for all $a, b \in R$
3. $\theta(1) = 1$

From this definition and the ring axioms also follows:

- $\theta(0) = 0$
- $\theta(-a) = -\theta(a)$

Notice that in the definition the operator $+$ and $\cdot$ is used in the ring R and also in ring R' depending on the context.
This definition just extends the definition of a homomorphism by the third property, $\theta(1) = 1$. This suffices to ensure that important properties are preserved under the homomorphism, such as $\theta(0) = 0$ and $\theta(-a) = -\theta(a)$.
Ring homomorphisms are a very abstract construct and can be found in many different domains. For our purposes, we will now consider two special homomorphisms, the modular homomorphism and the evaluation homomorphism.

Example 2 (modular homomorphism)*.* The homomorphism

$$\theta_m : \mathbb{Z}[x_1, \ldots, x_v] \to \mathbb{Z}_m[x_1, \ldots, x_v]$$

is defined for a fixed $m \in \mathbb{Z}$ by:

- $\theta_m(x_i) = x_i \qquad \text{for } 1 \leq i \leq v$
- $\theta_m(a) = rem(a, m) \qquad \text{for all coefficients } a \in \mathbb{Z}$

$rem(a, m)$ is the function that returns the remainder of the division of a by m. Intuitively, this means replace all coefficients by their "modulo m" representation.

We examine the following polynomial with the homomorphism θ_5: For the polynomial $a(x, y) = 2xy + 7x - y^2 + 8 \in \mathbb{Z}[x, y]$:
$\theta_5(a) = 2xy + 2x - y^2 - 2 \in \mathbb{Z}_5[x, y]$

The modular homomorphism obviously reduces the absolute value of all coefficients to bounds depending on m.
The next homomorphism is the evaluation homomorphism:

Example 3 (evaluation homomorphism)*.* The homomorphism

$$\theta_{x_i - \alpha} : D[x_1, \ldots, x_v] \to D[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_v]$$

is defined for a particular indeterminate x_i and a fixed $\alpha \in D$ by:

$$\theta_{x_i-\alpha}(a(x_1,\ldots,x_v)) = a(x_1,\ldots,x_{i-1},\alpha,x_{i+1},\ldots,x_v)$$

Again intuively, this means substitute the value α for the indeterminate x_i.

For a polynomial $a(x,y) = 2xy + 7x + y^2 + 8 \in \mathbb{Z}[x,y]$, this results in $\theta_{x-2}(a) = 4y + 14 + y^2 + 8 \in \mathbb{Z}[y]$

With the evaluation homomorphism, we can reduce polynomials in the number of indeterminates.

In the following section, we will show that ring homomorphisms can be characterized by ideals. Therefore we define ideals, show that the kernel of every homomorphism is an ideal and the kernel of an homomorphism completely determines the homomorphic image (up to isomorphism).

Characterization of homomorphisms

Ring homomorphisms can be uniquely be characterized by ideals.

Definition 4. Let R be a commutative ring. A nonempty subset I of R is called ideal if

1. $a + b \in I$ for all $a, b \in I$
2. $-a \in I$ for all $a \in I$
3. $ar \in I$ for all $a \in I$ and for *all* $r \in R$.

The definition of an ideal implies that an ideal is closed under all ring operations, addition, multiplication and negation, but it even implies more than that. The third property says that it even closed under multiplication by any element of R.
To get a better insight of ideals, we will regard a few examples:

Example 5 (Examples for ideals). The following examples are all ideals:

- $\langle m \rangle \subset \mathbb{Z} = \{m \cdot r : r = 0, \pm 1, \pm 2, \ldots\}$
- $\langle 4 \rangle = \{0, \pm 4, \pm 8, \pm 12, \ldots\}$
- $\langle p(x) \rangle \subset \mathbb{Z}[x] = \{p(x) \cdot a(x) : a(x) \in \mathbb{Z}[x]\}$
- $\langle x - 2 \rangle = \{(x-2) \cdot a(x) : a(x) \in \mathbb{Z}[x]\}$

We can easily see that the properties for ideals hold for these four examples.
Ring homomorphism can be characterized by ideals because of the following reason:

Correspondence of ideals and homomorphisms

We note that:

- Let R and R' be commutative rings. The kernel K of a homomorphism $\theta : R \to R'$ is an ideal in R.
- If $\theta_1 : R \to R'$ and $\theta_2 : R \to R''$ have the kernel K, the two homomorphic images are $\theta_1(R)$ and $\theta_2(R)$ are isomorphic.
- Consequently, homomorphism can be constructed and notated using their ideal.
- Congruence Arithmetic can be done *modulo I* for any ideal I.

Given a homomorphism, we can determine the kernel which is the characterizing ideal. On the opposite, we can construct a homomorphism given an ideal which should serve as kernel.
Furthermore, there is a natural way of defining arithmetic for any ideal, which we annotate with "modulo I".

Example 6. Consider

- The homomorphism θ_4 has the kernel/ideal $\langle 4 \rangle$.
- The homomorphism θ_{x-2} has the kernel $\langle x-2 \rangle$.
- Evaluation of p(x): $p(c) = d$ is equivalent to $p(x) \equiv d \ mod \ (x-c)$.
- From an "ideal" viewpoint, modular and evaluation homomorphisms are the same.

From these examples, we can see that evaluation of a polynomial at a point c is isomorph to the operation "modulo (x - c)". Furthermore, we will now define operations on ideals:

Operations on ideals

- The ideal $\langle a_1, a_2, \ldots, a_n \rangle$ is defined as $\{a_1 r_1 + \cdots + a_n r_n : r_i \in R\}$ $a_1, \ldots, a_n \in R$ is called basis.
- For ideal I = $\langle a_1, \ldots, a_n \rangle$ and $J = \langle b_1, \ldots, b_m \rangle$: The sum of two ideals is

  $$I + J = \langle I + J \rangle = \langle I, J \rangle = \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle$$

  The product of two ideals is

  $$I \cdot J = \langle I \cdot J \rangle = \langle a_1 b_1, \ldots, a_1 b_m, a_2 b_1, \ldots, a_2 b_m, \ldots, a_n b_1, \ldots, a_n b_m \rangle$$

  The i-th power is recursively defined by:

  $$I^1 = I \text{ and } I^i = I \cdot I^{i-1} \text{ for } i \geq 2.$$

Example 7. The following examples give you a simple and informal explanation how the elements in the presented ideals look like:

- $\langle x, y \rangle$ are all polynomials $a_1 x + a_2 y$ with $a_1, a_2 \in R[x, y]$.
- $\langle x, y \rangle \cdot \langle x, y \rangle$ are all polynomials $a_1 x^2 + a_2 xy + a_3 y^2$ with $a_1, a_2, a_3 \in R[x, y]$.
- $\langle x, y \rangle^k$ are all polynomials with terms of at least a total degree k.

Now, we have considered enough basics and we can get started with inverting the homomorphisms. First, we will look at simple inversion methods, the Chinese Remainder Algorithm and the Newton Interpolation.

2 Chinese Remainder Algorithm and Newton Interpolation

The Chinese Remainder Algorithm, also known as Garner's Algorithm, can invert modular homomorphisms by using the information of multiple modular homomorphisms to construct the integer number which the homomorphism were applied to. We assume that the integer numbers are all bounded by a fixed number and the homomorphisms are suitable to uniquely determine the integer number from the image problems.

Inverting modular homomorphisms with Chinese Remainder Algorithm

The Chinese Remainder problem is stated as follows: Given pairwise comaximal ideals $I_0, I_1, \ldots, I_n$ and given corresponding residues $s_i \in \mathbb{Z}/I_i$, $0 \le i \le n$, find an integer $u \in \mathbb{Z}/\prod_{i=0}^{n} I_i$ such that

$$u \equiv s_i \ mod\ I_i, 0 \le i \le n.$$

The Chinese Remainder Algorithm: Garner's Algorithm

The key to the algorithm: Express the solution $u \in \mathbb{Z}/\prod_{i=0}^{n} I_i$ in mixed radix representation.

Definition 8 (mixed radix representation)**.** A element $u \in Z/\prod_{i=0}^{n} I_i$ is in mixed radix representation when it is in the form

$$u^{(1)} + \Delta u^{(1)} + \Delta u^{(2)} + \ldots + \Delta u^{(n)}$$

where $u^{(1)} = u_0 \in Z/I_0$ and $\Delta u^{(k)} \in \prod_{i=0}^{k-1} I_i / \prod_{i=0}^{k} I_i$ for $1 \le k \le d$ and n is the number of equations.

We define $u^{(k+1)} = u^{(1)} + \Delta u^{(1)} + \ldots + \Delta u^{(k)}$.

Mixed radix representation

So, for ideals $I_i = \langle m_i \rangle$, the elements $\Delta u^{(k)}$ can be represented in the following form:

$$\Delta u^{(k)} = u_k \cdot \prod_{i=0}^{k-1} m_i$$

where $u_k \in Z_{m_k}$ for $0 \le k \le n$. Therefore, u can be written as:

$$u = u_0 + u_1 \cdot m_0 + u_2 \cdot (m_0 m_1) + \cdots + u_n \cdot (\prod_{i=0}^{n-1} m_i)$$

Example 9. We would like to represent the number five in mixed radix representation for $m_0 = 3; m_1 = 5; m = 3 \cdot 5 = 15$

$5 = (-1) + 2 \cdot 3$

We can see that any number from -7 to 7 can be represented in this form.

From modulo equations to mixed radix form

To construct the mixed radix form from the modulo equation, we just iterate over the modular equations and calculate one new element u_i from the modular equation and the existing elements from the prior iterations. The algorithm is scetched as follows: Iteration over $i = 0 \ldots n$:

- For $i = 0$: $u = s_0 \ mod\ m_0$ Choose $u_0 = s_0$.
- For $i = k$: $u_0, \ldots, u_{k-1}$ are known.

  Solve $u_0 + u_1(m_0) + u_2(m_0 m_1) + \cdots + u_k(\prod_{i=0}^{k-1} m_i) \equiv s_k \ mod\ m_k$

$$\Longrightarrow u_k \equiv \left(s_k - \left(u_0 + \cdots + u_{k-1}\left(\prod_{i=0}^{k-2} m_i\right)\right)\right)\left(\prod_{i=0}^{k-1} m_i\right)^{-1} \ mod\ m_k$$

We recall that we get from mixed radix representation to standard representation by evaluation with Horner scheme.

Uniqueness of the Chinese Remainder problem

The Chinese Remainder Problem can be uniquely solved and can be transfered from the domain $\mathbb{Z}/\prod_{i=0}^{n} I_i$ to $\mathbb{Z}$ if all moduli $m_0, \ldots, m_n$ are pairwise prime and $a \leq u \leq a+m$ with $m = \prod_{i=0}^{n} m_i$ for any fixed integer $a \in \mathbb{Z}$.

Inverting evaluation homomorphisms with Newton Interpolation

The polynomial interpolation problem is stated as follows: Let D be a domain of polynomials over a coefficient field R. Given ideals $\langle x-\alpha_0\rangle, \langle x-\alpha_1\rangle, \ldots, \langle x-\alpha_n\rangle$ where $\alpha_i \in R, 0 \leq i \leq n$ and given corresponding residues $s_i \in D, 0 \leq i \leq n$, find a polynomial $u(x) \in D[x]$ such that

$$u(x) \equiv s_i \ mod \ x - \alpha_i, 0 \leq i \leq n.$$

$\alpha_1, \ldots, \alpha_n$ are also called interpolation points. The polynomial interpolation problem can be uniquely solved with Newton interpolation if $deg(u(x)) \leq n$ with $n+1$ distinct interpolation points.

From a viewpoint of ideals, modular and evaluation homomorphisms are quite similar. Therefore the Garner's algorithm and the Newton interpolation algorithm are also very similar.

3 The Hensel lifting

In this section, we look at the Hensel lifting for the factorization problem.

The factorization problem

We consider the following problem: Given a polynomial a(x), we look for two polynomials u(x), w(x) such that

$$a(x) = u(x) \cdot w(x)$$

Reformulated, we are looking for a root of the function

$$F(u, w) = a(x) - u(x)w(x)$$

Assume, we found a solution $u^{(1)}$ and $w^{(1)}$ in R/I. We now invert a homomorphism $\theta_I : R \to R/I$ lifting two polynomials u and v as solution by an iterative method. This iterative method is called the Hensel construction. Analogous to the Chinese Remainder Algorithm, we find better approximation step by step to our solution. Therefore, we define:

Ideal-adic representation

Definition 10. Let I be an given ideal. A polynomial u is in ideal-adic representation when it is in the form

$u^{(1)} + \Delta u^{(1)} + \Delta u^{(2)} + \ldots + \Delta u^{(d)}$ where $u^{(1)} \in R/I$ and $\Delta u^{(k)} \in I^k/I^{k+1}$

for $1 \leq k \leq d$ and d is maximal total degree of u with respect to I.

We define $u^{(k+1)} = u^{(1)} + \Delta u^{(1)} + \ldots + \Delta u^{(k)}$.

Ideal-adic approximation

Definition 11. Let $I \subset R$ be an ideal. For a given polynomial $a \in R$, a polynomial $b \in R$ is an order k ideal-adic approximation to a with respect to I if

$$a \equiv b \ mod \ I^k$$

The error approximating a by b is $a - b \in I^k$.

Example 12. The polynomial $u^{(k)}$ is an order k ideal-adic approximation to the polynomial u.

The iteration step of the Hensel construction

- Assume, we already have a pair of order k approximations $u^{(k)}$ and $w^{(k)}$, so $F(u^{(k)}, w^{(k)}) \equiv 0 \ mod \ I^k$.
- We want to get k+1 order approximations $u^{(k+1)} = u^{(k)} + \Delta u^{(k)}$ and $w^{(k+1)} = w^{(k)} + \Delta w^{(k)}$
- So $F(u^{(k)} + \Delta u^{(k)}, w^{(k)} + \Delta w^{(k)}) \equiv 0 \ mod \ I^{k+1}$
- $F(u^{(k)}, w^{(k)}) + \frac{\delta F}{\delta u}(u^{(k)}, w^{(k)})\Delta u^{(k)} + \frac{\delta F}{\delta w}(u^{(k)}, w^{(k)})\Delta w^{(k)} \equiv 0 \ mod \ I^{k+1}$
- With $F(u, w) = a - uw$, we get: $F(u^{(k)}, w^{(k)}) - w^{(k)}\Delta u^{(k)} - u^{(k)}\Delta w^{(k)} \equiv 0 \ mod \ I^{k+1}$
- Finally, we have: $w^{(k)}\Delta u^{(k)} + u^{(k)}\Delta w^{(k)} \equiv F(u^{(k)}, w^{(k)}) \ mod \ I^{k+1}$

The final result states that to improve our k order approximation to a $(k+1)$ order approximation we must solve equations of the following form in each iteration:

$$w^{(k)}\Delta u^{(k)} + u^{(k)}\Delta w^{(k)} \equiv F(u^{(k)}, w^{(k)}) \ mod \ I^{k+1}$$

In which form the variables $w^{(k)}, \Delta u^{(k)}, u^{(k)}, \Delta w^{(k)}$ occur depends on the homomorphism to invert. For inverting a modular homomorphism, we can use the univariate Hensel lifting. To invert a evaluation homomorphism, we need to use the multivariate Hensel lifting. So in the following of this section, we will consider the univariate Hensel lifting and the multivariate Hensel lifting.

Univariate Hensel lifting

We want to invert a modular homomorphism $\theta_p : \mathbb{Z}[x] \to \mathbb{Z}_p[x]$. So given polynomials $a(x) \in \mathbb{Z}[x]$ and $u^{(1)}(x), w^{(1)}(x) \in \mathbb{Z}_p[x]$ such that

$$a(x) \equiv u_0(x)w_0(x) \ mod \ p$$

calculate $u(x), w(x) \in \mathbb{Z}_{p^t}[x]$ such that

$$F(u, w) = a(x) - uw = 0 \text{ and } u(x) \equiv u^{(1)}(x) \ mod \ p \text{ and }$$
$$w(x) \equiv w^{(1)}(x) \ mod \ p$$

The general ideal-adic representation and approximation simplifies to the p-adic representation and approximation.

p-adic representation and approximation

Definition 13. Let $I = \langle p \rangle$ be an ideal and let $R = \mathbb{Z}[x]$. A polynomial $u(x)$ is in its polynomial p-adic representation when it is in the form

$u^{(1)} + \Delta u^{(1)} + \Delta u^{(2)} + \ldots + \Delta u^{(d)}$ where $u^{(1)} \in R/I$ and $\Delta u^{(k)} \in I^k/I^{k+1}$

for $1 \leq k \leq d$ and d is maximal total degree of u with respect to I.

We define $u^{(k+1)} = u^{(1)} + \Delta u^{(1)} + \ldots + \Delta u^{(k)}$.

More specific view at the p-adic representation

$u^{(1)} \in R/I$ is a polynomial with coefficients in $\mathbb{Z}/p$. and the elements $\Delta u^{(k)}$ can be represented in the following form:

$$\Delta u^{(k)} = u_k(x) \cdot p^k$$

where $u_k \in \mathbb{Z}_{p^k}[x]$ for $0 \leq k \leq n$. Therefore, u can be written as:

$$u(x) = u_0(x) + u_1(x)p + u_2(x)p^2 + \cdots + u_n(x)p^n.$$

order n p-adic approximation

Definition 14. Let $a(x) \in \mathbb{Z}[x]$ be a given polynomial. A polynomial $b(x) \in \mathbb{Z}[x]$ is called an order n p-adic approximation to a(x) if

$$a(x) \equiv b(x) \ mod \ p^n$$

The error in approximating a(x) by b(x) is $a(x) - b(x) \in \mathbb{Z}[x]$.

Example 15. $u(x) = 27x^2 + 11x + 7$ in polynomial p-adic representation for $p = 5$: $u(x) = (2x^2 + x + 2) + (2x + 1) \cdot 5 + x^2 \cdot 5^2$

Now that we have defined our representation and approximation, we can reconsider the iteration step of the Hensel lifting.

The iteration step of the Hensel lifting

- We have order k approximations to $u(x)$ and $w(x)$, called $u^{(k)}$ and $w^{(k)}$.
- Recall that $w^{(k)}\Delta u^{(k)} + u^{(k)}\Delta w^{(k)} \equiv F(u^{(k)}, w^{(k)}) \ mod \ I^{k+1}$
- Solve $w_0(x)u_k(x) + u_0(x)w_k(x) = \theta_p \left(\frac{a(x) - u^{(k)}w^{(k)}}{p^k} \right)$ with Extended Euclidean Algorithm
- Define $u^{(k+1)} = u^{(k)} + u_k(x)p^k$ and $w^{(k+1)} = w^{(k)} + w_k(x)p^k$ and repeat iteration.

For the univariate Hensel lifting, we do one Extended Euclidean calculation for every iteration step.

Uniqueness of the Hensel construction

If $a(x) \in \mathbb{Z}[x]$ is monic and $u^{(1)}$ and $w^{(1)}$ are monic and relative prime, then there are uniquely determined monic polynomials factors $u^{(k)}$ and $w^{(k)}$ for any $k \geq 1$. For a non-monic polynomial a(x), some pre- and postprocessing has to be done.

Example for univariate Hensel lifting

- Factorizing $a(x) = x^3 + 10x^2 - 432x + 5040$ with $p = 5$
- Applying $\theta_5(a(x)) = x^3 - 2x = x(x^2 - 2) = u^{(1)} \cdot w^{(1)}$
- First iteration of Hensel construction
  - Calculate $\theta_5(\frac{a(x) - u^{(1)} w^{(1)}}{5}) = 2x^2 - x - 2$
  - Solve $(x^2 - 2)u_1(x) + xw_1(x) = 2x^2 - x - 2$
  - $u_1(x) = 1; w_1(x) = x - 1$
  - $u^{(2)} = u^{(1)} + u_1(x) \cdot p = x + 5$
    $w^{(2)} = w^{(1)} + w_1(x) \cdot p = x^2 + 5x - 7$
- Next iterations:

Iter	u_k	w_k	$u^{(k)}(x)$	$w^{(k)}(x)$	$e(x)$
0	-	-	x	$x^2 - 2$	$10x^2 - 430x + 5040$
1	1	$x - 1$	$x + 5$	$x^2 + 5x - 7$	$-450x + 5075$
2	1	$-x + 2$	$x + 30$	$x^2 - 20x + 43$	$125x + 3750$
3	0	1	$x + 30$	$x^2 - 20x + 168$	0

Whereas the iteration step of the univariate Hensel lifting is still simple (by using the Extended Euklidean Algorithm once), the iteration step of the multivariate Hensel lifting will turn out to be more difficult - even if it is conceptual the same.

4 Multivariate Hensel lifting

For the multivariate Hensel lifting we are interested in inverting a multivariate evaluation homomorphism $\theta_I : \mathbb{Z}[x_1, \ldots, x_v] \to \mathbb{Z}[x_1]$. So given polynomials $a(x) \in \mathbb{Z}[x_1, \ldots, x_v]$ and $u^{(1)}(x_1), w^{(1)}(x_1) \in R/I$ such that

$$a(x_1) \equiv u^{(1)}(x_1)w^{(1)}(x_1) \ mod\ I$$

calculate $u(x_1, \ldots, x_v), w(x_1, \ldots, x_v) \in R[x_1, \ldots, x_v]$ such that

$$F(u, w) = a(x) - uw = 0 \text{ and } u(x_1, \ldots, x_v) \equiv u^{(1)}(x_1) \ mod\ I \text{ and}$$
$$w(x_1, \ldots, x_v) \equiv w^{(1)}(x_1) \ mod\ I$$

The ideal I has the form $\langle x_2 - \alpha_2, \ldots, x_v - \alpha_v \rangle$.

Ideal-adic representation

Analogously to p-adic representation, we can define a ideal-adic representation for an ideal I.

Definition 16. Let $I = \langle x_2 - \alpha_2, x_3 - \alpha_3, \ldots, x_v - \alpha_v \rangle$ be an given ideal. A polynomial $u(x_1, \ldots, x_v)$ is in ideal-adic representation when it is in the form

$u^{(1)} + \Delta u^{(1)} + \Delta u^{(2)} + \ldots + \Delta u^{(d)}$ where $u^{(1)} \in R/I$ and $\Delta u^{(k)} \in I^k/I^{k+1}$

for $1 \leq k \leq d$ and d is maximal total degree of u with respect to I.

We define $u^{(k+1)} = u^{(1)} + \Delta u^{(1)} + \ldots + \Delta u^{(k)}$.

More specific view at the ideal-adic representation

The term $u^{(1)}$ is $u(x_1, \alpha_2, \alpha_3, \ldots, \alpha_v)$. A term $\Delta u^{(k)} \in I^k$ is a sum of all terms with total degree of k with respect to I, so it has the form

$$\underbrace{\sum_{i_1=2}^{v} \sum_{i_2=i_1}^{v} \cdots \sum_{i_k=i_{k-1}}^{v}}_{k\ sums} \underbrace{u_i^{(k)}(x_1)}_{coefficient} \underbrace{(x_{i_1} - \alpha_{i_1}) \cdot (x_{i_2} - \alpha_{i_2}) \cdot \ \cdots \ \cdot (x_{i_k} - \alpha_{i_k})}_{k\ factors}$$

where $2 \leq i_1 \leq \ldots \leq i_k \leq v$ and i is a vector with k entries of indices $= (i_1, i_2, \ldots, i_k)$

Ideal-adic approximation

Definition 17. Let I be an ideal in $\mathbb{Z}[x_1, \ldots, x_v]$. For a given polynomial $a \in \mathbb{Z}[x_1, \ldots, x_v]$, a polynomial $b \in \mathbb{Z}[x_1, \ldots, x_v]$ is an order k ideal-adic approximation to a with respect to I if

$$a \equiv b \ mod \ I^k$$

The error is approximating a by b is $a - b \in I^k$.

Example 18. The polynomial $u^{(k)}$ is an order k ideal-adic approximation to the polynomial u.

Iteration step for multivariate Hensel construction

From an k order ideal-adic approximation $u^{(k)}$ and $w^{(k)}$, we calculate an k+1 order ideal-adic $u^{(k+1)}$ and $w^{(k+1)}$ approximation.

- The update formula

$$w^{(k)}\Delta u^{(k)} + u^{(k)}\Delta w^{(k)} = (a(x_1, \ldots, x_v) - u^{(k)}w^{(k)}) \ mod \ I^{k+1}$$

- Represent

$$a(x_1, \ldots, x_v) - u^{(k)}w^{(k)} =$$
$$\textstyle\sum_{i_1=2}^{v} \sum_{i_2=i_1}^{v} \cdots \sum_{i_k=i_{k-1}}^{v} c_i^{(k)}(x_1)(x_{i_1} - \alpha_{i_1}) \cdot \ \cdots \ \cdot (x_{i_k} - \alpha_{i_k})$$

- Separate and simplify equation to $w^{(1)}u_i(x_1) + u^{(1)}w_i(x_1) = c_i(x_1)$
- Solve with Extended Euclidean Algorithm

The idea of the multivariate Hensel construction is the same as in the univariate case. The calculation has become more inscrutable because the equations now contain k nested summations for the k-th iteration step. Whereas this is not technically difficult to split up to different equations and solve, we cannot present any simple example as the number of equations is just too large.

Outlook

We did not discuss

- Leading Coefficient Problem in the univariate Hensel construction
- Bad performance because of the Bad-Zero Problem
- Using sparseness of solution to improve Hensel construction

- Quadratic Iteration, also known as Zassenhaus construction

This paper is intended to give you an general insight of the Hensel construction. An implementation following the idea of this paper would be possible, but the execution time would be still too large for pratical purposes, because many improvements would have to added to get an useful algorithm. The most important improvement is taking advantage of the sparseness of solutions, which primarily justifies the Hensel construction instead of using the Chinese Remainder Algorithm. Many details have not been presented, but the uncovered topics can be found in the following literature.

References

[GCL92] Keith O. Geddes, Stephen R. Czapor, and George Labahn. *Algorithms for computer algebra*. Kluwer Academic Publishers, Norwell, MA, USA, 1992.

[MY74] Alfonso Miola and David Y. Y. Yun. Computational aspects of Hensel-type univariate polynomial greatest common divisor algorithms. 8(3):46–54, August 1974.

[Yun74] D. Y. Y. Yun. *The Hensel Lemma in algebraic manipulation*. PhD thesis, M.I.T. , Reprint Garland Publ. NY, 1980, 1974.

[Zip81] Richard Zippel. Newton's iteration and the sparse hensel algorithm (extended abstract). In *SYMSAC '81: Proceedings of the fourth ACM symposium on Symbolic and algebraic computation*, pages 68–72, New York, NY, USA, 1981. ACM Press.