A very short introduction to the Finite Element Method

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1 Introduction

This is a short introduction to the finite element method (FEM), which is, besides others like the finite differences approximation, a technique to solve partial differential equations (PDE's) numerically.

The FEM was mainly developed for equations of elasticity and structural mechanics. In these fields problems have to be solved in complicated and irregular geometries. So one of the main advantages of the FEM, in comparison to the finite differences approximation, lies in the flexibility concerning the geometry of the domain where the PDE is to be solved. Moreover the FEM is perfectly suitable as adaptive method, because it makes local refinements of the solution easily possible.

The method does not operate on the PDE itself, instead the PDE is transformed to a equivalent variational or weak form. This will be the topic of the second part: the variational principle.

A first approach to solve the variational or weak form was made by Ritz (1908). A discussion of this method is the subject in the third section: the Ritz method.

Considering the disadvantages of the Ritz method will lead to the finite element method and to the fourth and last part: the finite element method.

2 The Variational Principle

Partial differential equations (PDE) are separated into different types which behave very differently and demand an entirely own treatment. In the field of second-order linear differential equations three types are of fundamental interest, these are the hyperbolic, parabolic and elliptic equations. Depending on the type of the PDE boundary or initial conditions have to be given.

The main focus of the finite element method are elliptic PDE's so we will concentrate on this type. The correct side conditions for elliptic PDE's are boundary conditions.

Definition: Elliptic Partial Differential Equation
Let be $\Omega \in \mathbb{R}^n$ open, $f : \Omega \to \mathbb{R}$ and L a linear elliptic Operator in the form

$$Lu = -\sum_{i,j=1}^{n} \partial_j(a_{ij}(x)\,\partial_i u) + a_0(x)u \tag{1}$$

where $A(x) := (a_{ij}(x))_{ij}$ symmetric positive definite.
Then

$$Lu = f \quad \text{in } \Omega$$

is called elliptic partial differential equation.

Example 1:

$$\text{for} \quad A(x) = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \Leftrightarrow a_{ij}(x) = \delta_{ij} \quad \text{and} \quad a_0(x) = 0$$

$$\Rightarrow Lu = -\sum_{i=1}^{n} \partial_i^2 u = -\Delta u$$

so we obtain the Poisson Equation $-\Delta u = f$ which is the prototype of an elliptic partial differential equation.

Boundary conditions for elliptic partial differential equations are

$$\begin{array}{lrll} & u = g & \text{on } \partial\Omega & \text{(Dirichlet boundary condition)} \\ \text{or} & \partial_\nu u = h & \text{on } \partial\Omega & \text{(Neumann boundary condition)} \end{array}$$

Consider now the elliptic PDE with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} Lu &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

for this situation will we state the following theorem.

Theorem: The Variational Principle
Let be L a linear elliptic Operator as in (1), $a_0 \geq 0$ and $V = \{v \in C^2 \cap C(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega\}$.
For $u \in V$ are equivalent:

(i) $Lu = f$

(ii) $a(u,v) = (f,v) \qquad \forall v \in V$
where $a(u,v) = \int\limits_\Omega \sum_{i,j=1}^n a_{ij}\partial_i u \partial_j v + a_0 uv dx$
and $(f,v) = \int\limits_\Omega fv dx$

(iii) $u = \arg\min\limits_{v \in V} I(v) \qquad$ *with* $I(v) = \frac{1}{2}a(v,v) - (f,v)$

Proof:

(i)⇔(ii):

$$\begin{aligned} Lu = f \quad &\Leftrightarrow \quad 0 = \int\limits_\Omega v(Lu - f)dx \qquad \forall v \in V \\ &\Leftrightarrow \quad 0 = \int\limits_\Omega \sum_{i,j}^n a_{ij}\partial_i u \partial_j v + a_0 uv - fv dx \qquad \forall v \in V \\ &\Leftrightarrow \quad a(u,v) = (f,v) \qquad \forall v \in V \end{aligned}$$

the second transformation is done by Green's Theorem whereas the boundary integral is zero since $v \in V$ vanishes on $\partial\Omega$.

(ii)⇒(iii): For $u, v \in V$ and $t \in \mathbb{R}$ we get

$$\begin{aligned} I(u+tv) &= \frac{1}{2}a(u+tv, u+tv) - (f, u+tv) \\ &= I(u) + t(\underbrace{a(u,v) - (f,v)}_{=0}) + \frac{t^2}{2}a(v,v) \end{aligned}$$

If $t = 1$ and $v \neq 0$ then

$$\begin{aligned} I(u+v) &= I(u) + \frac{1}{2}\underbrace{a(v,v)}_{>0} \\ &> I(u) \end{aligned}$$

So u minimises I over V. Additionally, for $v \in V$ is: $a(v,v) = 0 \Rightarrow v = 0$, thus u is the unique minimal point of I over V.

(iii)⇒(ii): I has a stationary point at u, since u minimises I, so

$$\frac{\partial}{\partial t} I(u+tv)\big|_{t=0} = 0$$

with

$$I(u+tv) = \frac{1}{2}a(u,u) - (f,u) + ta(u,v) - (f,v) + t^2\frac{1}{2}a(v,v)$$

we recieve

$$\begin{aligned}\frac{\partial}{\partial t}I(u+tv)\big|_{t=0} &= (a(u,v) - (f,v) + t(a(v,v))\big|_{t=0} \\ &= a(u,v) - (f,v)\end{aligned}$$

together

$$0 = a(u,v) - (f,v)$$

Remarks:

- The boundary condition $u = 0$ on $\partial\Omega$ is satisfied through the choice of the space $V = \{v \in C^2(\Omega) \cap C(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega\}$.
- Equation (ii) $a(u,v) = (f,v)$ is called weak form of the pde.
- Equation (iii) $u = \arg\min_{v\in V} I(v)$ is called variational form of the pde.
- If such an u exists, it is unique.
- A solution can be guaranteed (by the *Lax-Milgram Theorem*) if the space V is expanded appropriate ($\rightarrow$ *Sobolev Spaces*)

Example 2: Assume the Poisson equation with homogeneous Dirichlet boundary condition:

$$-\Delta u = f \qquad u = 0 \text{ on } \partial\Omega \tag{2}$$

With $V = \{v \in C^2(\Omega) \cap C(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega\}$ and $a(u,v) = \int_\Omega \sum_{i,j} a_{ij}\partial_i u \partial_j v + a_0 uv dx$ where $a_{ij} = \delta_{ij}, a_0 = 0$ (cp. example 1) so $a(u,v) = \int_\Omega \sum_{i,j} \partial_i u \partial_j v dx$

the variational principle says, that instead of (2) we may solve

$$\min_{v\in V} I(v) = \frac{1}{2}\underbrace{\int_\Omega \sum_i (\partial_i)^2 dx}_{a(v,v)} - \underbrace{\int_\Omega f v dx}_{(f,v)}$$

or

$$a(u,v) = (f,v) \qquad \forall v \in V$$

So far we have been assuming that the problem is given with homogenous **Dirichlet boundary conditions** (i.e. $u = 0$ on $\partial\Omega$). In general this

may not be the case. Nonetheless it is no loss of generality if we assume homogenous Dirichlet boundary conditions.

Suppose the problem: $Lu = f, \quad u = g$ on $\partial\Omega$
If we solve the equation $a(w, v) = (f - Lg, v) \quad \forall v \in V$

$$\begin{aligned} \text{we get with } w := u - g \quad : \quad & a(w, v) = (f - Lg, v) \\ \Leftrightarrow \quad & a(u, v) - a(g, v) = (f, v) - (Lg, v) \\ \Leftrightarrow \quad & a(u, v) = (f, v) \end{aligned}$$

so u satisfies $Lu = f$. On $\partial\Omega$ we have:

$$0 = w\big|_{\partial\Omega} = (u - g)\big|_{\partial\Omega} \quad \Leftrightarrow \quad u\big|_{\partial\Omega} = g\big|_{\partial\Omega}$$

so u also satisfies the boundary condition $u = g$ on $\partial\Omega$.

Now we are going to have a short view on **Neumann boundary conditions**. Suppose the problem: $Lu = f, \quad \sum_{i,j}^{n} a_{ij}\partial_\nu u = h$ on $\partial\Omega$

Remember that we used Green's Theorem to transform (Lu, v) into $a(u, v)$, and no boundary integrals are left in the bilinear form a because of the homogeneous Dirichlet boundary condition ($v = 0$ on $\partial\Omega$). In the above problem Neumann boundary conditions are given, so we cannot expect that v will vanish at the boundary of Ω. Hence the boundary integral does not vanish and (Lu, v) becomes:

$$(Lu, v) = a(u, v) + (h, v)_{\partial\Omega} \quad \text{where } (h, v)_{\partial\Omega} = \int_{\partial\Omega} hv dx$$

so the weak form of the PDE changes to:

$$a(u, v) = (f, v) - (h, v)_{\partial\Omega}$$

and the variational form becomes:

$$I(v) = \frac{1}{2}a(v, v) - (f, v) - (h, v)_{\partial\Omega}$$

3 The Ritz Method

So far we have seen, that instead of solving the PDE we may solve the equation $a(u, v) = (f, v) \quad \forall v \in V$ with a particular bilinear form a which depends on the given PDE, or minimise a certain functional $I(v) = \frac{1}{2}a(v, v) - (f, v)$ respectively.
But since $V = \{v \in C^2(\Omega) \cap C(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega\}$ is an infinite-dimensional space, this is still a though problem.

In 1908 Ritz posed the idea, to search the solution u on a finite-dimensional subspace $V_h \subset V$. So he was looking for a approximation of u by finding a function $u_h \in V_h$ that satisfies

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

This rises immediately the questions how to find such u_h, if it is easy to find it and how good it approximates our solution u. We will answer these questions in this order.

How to find this u_h? Let $(\varphi_i)_{i=1=n}$ be a basis of V_h. So $u_h \in V_h$ can be written as $u_h = \sum_{i=1}^{n} c_i \varphi_i$.
Then u_h must satisfy:

$$\begin{aligned}
& a(u_h, v_h) = (f, v_h) && \forall v_h \in V_h \\
\Leftrightarrow \quad & a(u_h, \varphi_j) = (f, \varphi_j) && \forall j \\
\Leftrightarrow \quad & a(\sum\nolimits_i c_i \varphi_i, \varphi_j) = (f, \varphi_j) && \forall j \\
\Leftrightarrow \quad & \sum\nolimits_i c_i \underbrace{a(\varphi_i, \varphi_j)}_{:=a_{ij}} = \underbrace{(f, \varphi_j)}_{:=b_j} && \forall j
\end{aligned}$$

which is nothing else but a, linear system of equations: $Ac = b$.

The approach from the variational form respectively the minimising problem

$$\begin{aligned}
\min_{v_h \in V_h} I(v_h) \quad & \Leftrightarrow \quad \frac{\partial}{\partial v_h} I(v_h) = 0 \qquad \forall v_h \in V_h \\
& \Leftrightarrow \quad \frac{\partial}{\partial c_j} I(\sum\nolimits_i c_i \varphi_i) = 0 \qquad \forall j
\end{aligned}$$

lead to the same linear system of equations.

Obviously this is now a very simple problem and many methods in numerical linear algebra exists to solve linear systems of equations, so finding u_h is very easy.

The Matrix $A = (a(\varphi_i, \varphi_j))_{j,i}$ is called **stiffness matrix** due to the fact that first applications of the finite element method has been made by engineers in the field of structural mechanics.

Example 3:
Consider the Poisson equation with homogeneous Dirichlet boundary conditions in one dimension:

$$-\Delta u = x^2, \quad u = 0 \text{ on } \partial\Omega \qquad \text{and } \Omega = [0, 1]$$

Let the basis $\{\varphi_1, \varphi_2\}$ of V_h be as follows:

$$\begin{aligned} \varphi_1 &= x(1-x) && \Rightarrow \quad \varphi_1' = 1 - 2x \\ \varphi_2 &= x^2(1-x) && \Rightarrow \quad \varphi_2' = 2x^2 - 3x^3 \end{aligned}$$

with $a(u,v) = \int_\Omega \sum_{i,j} \partial_i u \partial_j v dx$ (cp. example 2) we get

$$\begin{aligned} a(\varphi_1, \varphi_1) &= \int_0^1 (\varphi_1')^2 dx = \frac{1}{3} \\ a(\varphi_2, \varphi_2) &= \int_0^1 (\varphi_2')^2 dx = \frac{2}{15} \\ a(\varphi_1, \varphi_2) &= a(\varphi_2, \varphi_1) = \int_0^1 \varphi_1' \varphi_2' dx = \frac{1}{6} \end{aligned}$$

so the stiffness matrix is: $A_h = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{15} \end{pmatrix}$

Moreover

$$\begin{aligned} (f, \varphi_1) &= \int_0^1 f\varphi_1 dx = -\frac{1}{20} \\ (f, \varphi_2) &= \int_0^1 f\varphi_2 dx = -\frac{1}{30} \end{aligned}$$

therefore we obtain the right hand side $b_h = \begin{pmatrix} -\frac{1}{20} \\ -\frac{1}{30} \end{pmatrix}$.

Solving $A_h c = b_h$ yields $c = \begin{pmatrix} -\frac{1}{15} \\ -\frac{1}{6} \end{pmatrix}$.

Finally we receive the approximation u_h of u (Fig. 1):

$$u_h = c_1\varphi_1 + c_2\varphi_2 = -\frac{1}{6}x^3 + \frac{1}{10}x^2 + \frac{1}{15}x$$

Remarks:

- Solving a PDE is reduced to solving a linear system of equations, which is a relatively easy task in numerics and there are lot of methods for this problem

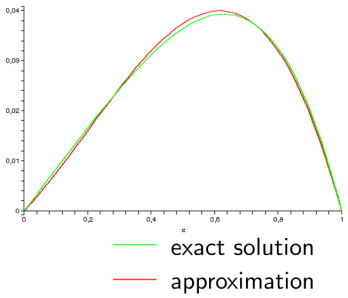

Figure 1: example for Ritz method

- u_h is the best approximation in V_h in respect to the Norm induced by a *(Cea Lemma)* (which answers our third questions raised in the beginning of this chapter).
- This method has still two disadvantages:
  First the matrix A_h is dense, so the computation of $A_h c = b_h$ will be relatively slow, because we can expect, that A_h will become very big for accurate approximations.
  And second may it be quite difficult to find a basis $(\varphi_i)_{i=1...n}$ of V_h where all the functions φ_i satisfy the boundary conditions, especially for irregular domains Ω.
  The finite element method will overcome these two problems.

4 The Finite Element Method

If it is possible to find a basis $(\varphi_i)_{i=1...n}$ of V_h where each φ_i vanishes on most part of the domain Ω (this is called φ_i has **local support** (Fig. 2)), it follows that:

$\rightarrow$ $a(\varphi_i, \varphi_j) = 0$ for most ij, because whenever one of the basis functions is zero at a certain point $x \in \Omega$ the product of φ_i' and φ_j' vanishes and so does $a(\varphi_i, \varphi_j)$.

$\rightarrow$ the matrix A_h will be sparse, since most of the entries $a(\varphi_i, \varphi_j)$ are zero.

$\rightarrow$ the boundary conditions have to be satisfied by only these few φ_i, which do not vanish at $\partial\Omega$.

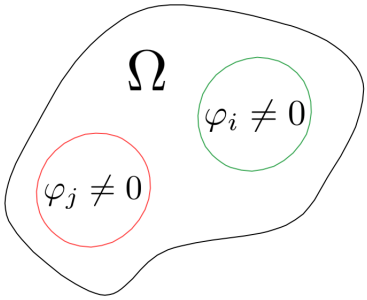

Figure 2: local support of basis functions

The idea of the FEM is to discretise the domain Ω into finite elements and define functions φ_i which vanish on most of these elements.

First we choose a geometric shape and divide the domain Ω into a finite number of regions. In one dimension the domain Ω is split into intervals. In two dimensions the elements are usually of triangular or quadrilateral shape. And in three dimensions tetrahedral or hexahedral forms are most common. Most elements used in practice have fairly simple geometries, because this results in very easy computation, since integrating over these shapes is quite basic.

The basis functions φ_i are usually not defined directly. Instead a function type, the so called ansatz function, (e.g. linear or quadratic polynomial) is selected which our approximation u_h of u should adopt on each of these elements. Most commonly a linear ansatz function is chosen, which means that u_h will be a linear function on each element and continuous over Ω (but not continuously differentiable).

Each element possesses a set of distinguishing points called nodal points or nodes. Nodes define the element geometry, and are the degrees of freedom of the ansatz function. So the number of nodes in a element depends on the ansatz function as well as the geometry. They are usually located at the corners or end points of elements. For higher-order (higher than linear) ansatz functions, nodes are also placed on sides or faces, as well as perhaps the interior of the element (Fig. 3).

The combination of the geometric shape of the finite element and their associated ansatz function on this region is referred as finite element type. The basis $(\varphi_i)_{i=1\ldots n}$ arise from the choice of the finite element type.

Example 4: Linear finite elements in 1 dimension
Let us approximate u by a piecewise linear function u_h on the domain $\Omega \subset \mathbb{R}$ (Fig. 4).

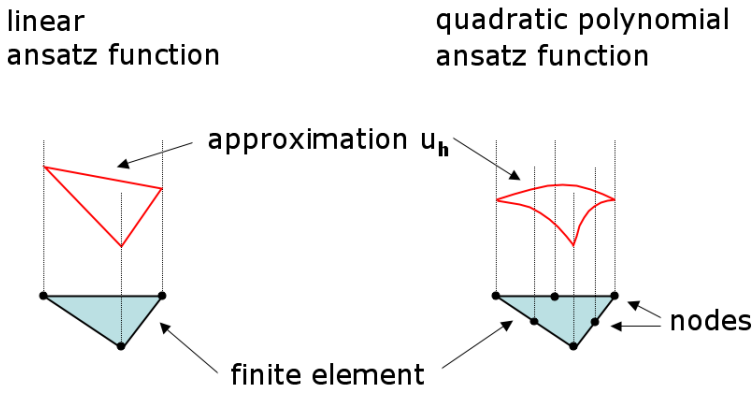

Figure 3: example of finite element types

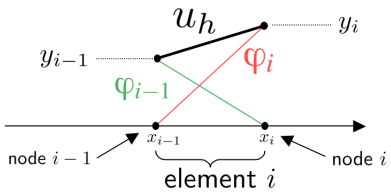

Figure 4: linear finite elements in 1 dimension

This leads to the well kmown B-Spline basis (Fig. 5):

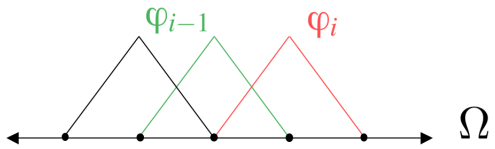

Figure 5: basis of 1 dimensional linear finite elements

$$\varphi_i(x) = \begin{cases} \frac{x-x_{i-1}}{x_i-x_{i-1}} & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & x \in [x_i, x_{i+1}] \\ 0 & else \end{cases}$$

And u_h can be written as

$$u_h = \sum_i y_i \varphi_i$$

In 2 dimensions the basis looks similar (Fig. 6).

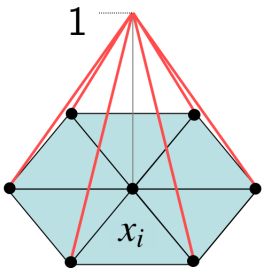

Figure 6: basis function of 2 dimensional linear finite elements

Advantages of the Finite Element Basis

- It is very easy to find a basis $(\varphi_i)_{i=1...n}$ for any given (and arbitrarily irregular) domain Ω.
- It is very easy to put up
- Every φ_i has local support

Model Algorithm of the Finite Element Method

1. transform the given PDE $Lu = f$ via the variational principle into $a(u, v) = (f, v)\ \forall v \in V$
2. select a finite element type
3. discretise the domain Ω
4. derive the basis $(\varphi_i)_{i=1\ldots n}$ from the discretisation and the chosen ansatz function
5. calculate the stiffness matrix $A_h = (a(\varphi_i, \varphi_j))_{ji}$ and the right hand side $b_h = (f, \varphi_i)$
6. solve $A_h c = b_h$
7. obtain (and visualise) the approximation $u_h = \sum_i c_i \varphi_i$

Remarks:

- The size of the stiffness matrix A_h and therefore the calculation costs depends on the number of nodes in the discretisation of Ω. Furthermore is the quantity of nodal points depending on the number of elements the domain Ω is divided into and the used ansatz function.
- Actually we do not have to put up the basis $(\varphi_i)_{i=1\ldots n}$ explicit (step 4) to calculate the stiffness matrix A_h. Instead it is possible to calculate for each element the contribution to A_h and sum these contributions up. Usually the contributions differ only by a factor from each other, so there are very few integrals to evaluate to acquire the stiffness matrix A_h.

Example 5:
Again, consider the Poisson equation with homogeneous Dirichlet boundary conditions in one dimension:

$$-\Delta u = x^2, \quad u = 0 \text{ on } \partial\Omega \qquad \text{and } \Omega = [0, 1]$$

We choose linear finite elements (Fig. 5) and with $a(u,v) = \int_\Omega \sum_{i,j} \partial_i u \partial_j v dx = \int_0^1 u'v'dx$ (cp. example 2) we get for each element (Fig. 7)

$$\begin{aligned} a(\varphi_i, \varphi_i) &= \int\limits_{element\ i} (\varphi_i')^2 dx = \frac{1}{h} \\ a(\varphi_{i-1}, \varphi_{i-1}) &= \frac{1}{h} \\ a(\varphi_{i-1}, \varphi_i) &= a(\varphi_{i-1}, \varphi_i) = \int\limits_{element\ i} \varphi_{i-1}' \varphi_i' dx = -\frac{1}{h} \end{aligned}$$

so the contribution from every element i to the stiffness matrix is

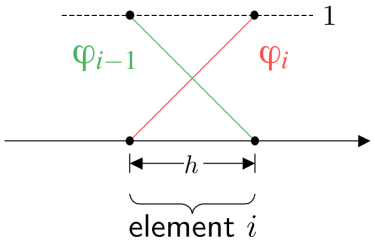

Figure 7: elementwise integration of the basis functions

$$A_{hi} = \frac{1}{h} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and under consideration of the boundary conditions we receive

$$A_h = \sum_i A_{hi} = \frac{1}{h} \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$$

Computing $b_h = (x^2, \varphi_i)$ is done similarly.
Solving $A_h c = b_h$ yields the approximation we see in Fig. 8.

This is definitely the most simple version of the FEM, but in principle all FEM programs work in this way.
More sophisticated versions of the FEM use geometric shapes which are more complicated and adjusted to the given problem. Moreover they use

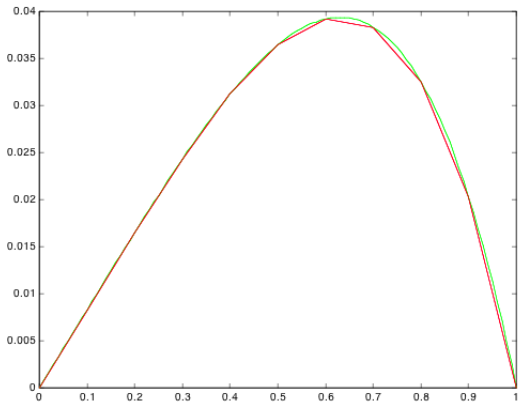

Figure 8: approximation of 1 dimensional poisson equation

other basis functions, to obtain approximations with certain properties, e.g. more smoothness.
Additionally it is usually possible to make local refinements of the approximation u_h which means that the resolution of u_h is not equal over the whole domain Ω. This leads to the situation, that the basis functions have a very small support in a certain region of the domain Ω (where a good approximation of u is important, e.g. the solution is changing very fast) and in other regions the support of the basis functions are relatively big.
But these are all subtleties, and this is only to draw the outline of the concept of the finite element method.

References

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[2] D. Braess: *Finite Elemente*, Springer-Verlag, 1997